NONLOCAL ELLIPTIC AND PARABOLIC PROBLEMS WITH SINGULAR AND CRITICAL NONLINEARITIES

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by

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To My Family

Certificate

This is to certify that the thesis entitled **”NONLOCAL ELLIPTIC AND PARABOLIC PROBLEMS WITH SINGULAR AND CRITICAL NONLINEARITIES”** submitted by **Ms. Tuhina Mukherjee** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

In this thesis, we study the existence and multiplicity of weak solutions of nonlocal problems involving fractional Laplace operator along with singular and critical nonlinearity. We begin with introducing briefly the fractional Laplacian, critical exponent problems, singular problems and Nehari manifold technique, which is subsequently used throughout the thesis.

The second chapter includes the study of existence and multiplicity of weak solutions of the following fractional singular equation

$$(P_\lambda^1): \qquad (-\Delta_p)^s u = \lambda u^{-q} + u^{\alpha}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where $\lambda > 0,\ s \in (0,1),\ 0 < q \leq 1,\ p < \alpha + 1 \leq p_s^*,\ p_s^* = \frac{np}{n-sp},\ n > sp$ and Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$. Using the fibering map analysis over the Nehari manifold, we show that there exists a $\Lambda_1 > 0$ such that (P_λ^1) admits at least two solutions when $\lambda \in (0, \Lambda_1)$. We have also studied the case $q \geq 1,\ \alpha = 2_s^* - 1$ case separately in the last section. Here, we use the results from non smooth analysis to show the multiplicity of weak solutions in L^1_{loc} for suitable range of λ. We also show regularity of the weak solutions obtained for this problem, that is they belong to $L^\infty(\Omega) \cap C^\alpha_{loc}(\Omega)$ where $\alpha \in (0,1)$.

The third chapter contains the existence and stabilization results for the following singular

parabolic equation involving the fractional Laplacian

$$(P_t^s)\begin{cases} u_t+(-\Delta)^s u = u^{-q}+f(x,u),\ u>0 \text{ in } \Lambda_T, \\ u=0 \text{ in } \Gamma_T, \\ u(0,x)=u_0(x) \text{ in } \mathbb{R}^n, \end{cases}$$

where $\Lambda_T=(0,T)\times\Omega$, $\Gamma_T=(0,T)\times(\mathbb{R}^n\setminus\Omega)$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ (at least C^2), $n>2s$, $s\in(0,1)$, $q>0$ such that $q(2s-1)<(2s+1)$ and $T>0$. The function $(x,y)\in\Omega\times\mathbb{R}\mapsto f(x,y)$ is assumed to be bounded from below Carathéodary function, locally Lipschitz with respect to the second variable and uniformly for $x\in\Omega$ and it satisfies

$$\limsup_{y\to+\infty}\frac{f(x,y)}{y}<\lambda_{1,s}(\Omega),$$

where $\lambda_{1,s}(\Omega)$ is the first eigenvalue of $(-\Delta)^s$ in Ω with Dirichlet boundary condition in $\mathbb{R}^n\setminus\Omega$. We prove the existence and uniqueness of a weak solution to (P_t^s) on assuming u_0 satisfies an appropriate cone condition. We use the semi-discretization in time with implicit Euler method and study the stationary problem to prove our results. We also show additional regularity on the solution of (P_t^s) when we regularize our initial function u_0.

We have proved a three solution theorem in the fourth chapter for a nonlocal singular problem. Precisely we considered the following nonlocal singular problem

$$(P_\lambda^2) \qquad (-\Delta)^s u=\lambda\frac{f(u)}{u^q},\ u>0 \text{ in } \Omega, \quad u=0 \text{ in } \mathbb{R}^n\setminus\Omega,$$

where $s\in(0,1)$, $q\in(0,1)$, $\lambda>0$ and Ω is a smooth bounded domain in $\mathbb{R}^n$. Here we assume$f:[0,\infty)\to[0,\infty)$ to be a continuous nondecreasing map satisfying $\lim\limits_{u\to\infty}\frac{f(u)}{u^{q+1}}=0$. Under certain additional assumptions on f, we prove that (P_λ^2) possesses at least three distinct solutions in a certain range of λ. We use the method of sub and supersolutions and a critical point theorem by Amann [8] to prove our results. Moreover we show that when λ becomes large enough, (P_λ^2) has a unique solution.

In the fifth chapter, we studied a doubly nonlocal Brezis-Nirenberg type Choquard equation

$$(P_\lambda^3) : (-\Delta)^s u = \lambda u + \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}y\right) |u|^{2^*_{\mu,s}-2}u \text{ in } \Omega, \;\; u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $\lambda \in \mathbb{R}$, $s \in (0,1)$, $2^*_{\mu,s} = (2n-\mu)/(n-2s)$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, $0 < \mu < n$ and $n > 2s$. Using the Mountain Pass theorem and Linking theorem we proved the existence of weak solution to (P_λ^3) depending on the range of the parameter λ. We also show that every solution is in $L^\infty(\Omega) \cap C^\alpha(\bar{\Omega})$ for some $\alpha > 0$ (depending on Ω and s) satisfying $\alpha < \min\{s, 1-s\}$. Moreover using the Pohozaev identity, we prove that when $\lambda < 0$ and Ω is strictly star shaped with respect to the origin then (P_λ^3) admits no positive nontrivial solution.

In the last chapter, we considered a system of equations involving the fractional Laplacian and critical Choquard nonlinearity that is

$$(P_{\lambda,\delta}) \begin{cases} (-\Delta)^s u = \lambda |u|^{q-2}u + \left(\displaystyle\int_\Omega \frac{|v(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \, \mathrm{d}y\right) |u|^{2^*_{\mu,s}-2}u \text{ in } \Omega, \\ (-\Delta)^s v = \delta |v|^{q-2}v + \left(\displaystyle\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \, \mathrm{d}y\right) |v|^{2^*_{\mu,s}-2}v \text{ in } \Omega, \\ \qquad u = v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > 2s$, $s \in (0,1)$, $\mu \in (0,n)$, $1 < q < 2$ and $\lambda, \delta > 0$ are real parameters. Using minimization over subsets of the Nehari manifold, we prove that the system $(P_{\lambda,\delta})$ admits at least two nontrivial solutions whenever $0 < \lambda^{\frac{2}{2-q}} + \delta^{\frac{2}{2-q}} < \Theta$ if $\mu \leq 4s$ and $0 < \lambda^{\frac{2}{2-q}} + \delta^{\frac{2}{2-q}} < \Theta_0$ if $\mu > 4s$, for some $\Theta, \Theta_0 > 0$ where possibly $\Theta_0 \leq \Theta$.

सार

इस थीसिस में, हम एकवचन और महत्वपूर्ण नॉनलिनियारिटि के साथ फ्रॅक्शनल लाप्लास ऑपरेटर शामिल नॉनलोकल समस्याओं के कमजोर समाधान के अस्तित्व और बहुतायत का अध्ययन करते हैं। हम संक्षेप में फ्रॅक्शनल लपलाशन, महत्वपूर्ण घातीय समस्याओं, एकवचन समस्याओं और नेहारी कई गुना तकनीक शुरू करने के साथ शुरू करते हैं, जिसे बाद में पूरे सिद्धांत में उपयोग किया जाता है ।

दूसरे अध्याय में निम्नलिखित अंशकालिक एकवचन समीकरण के कमजोर समाधानों के अस्तित्व और बहुतायत का अध्ययन शामिल है

(P_λ^1): $(-\Delta_p)^s u = \lambda u^{-q} + u^\alpha$, $u > 0$ Ω में, $u = 0$ $R^n \backslash \Omega$ में,

जहां $\lambda > 0$, $0<s<1$, $0 < q \leq 1$, $p < <\alpha + 1 \leq p^*_s$, $p^*_s = \frac{np}{n-sp}$, $n > sp$ और Ω R^n में चिकनी सीमा वाला एक बाध्य डोमेन है। नेहारी कई गुना पर फाइबरिंग मानचित्र विश्लेषण का उपयोग करते हुए, हम दिखाते हैं कि एक $\Lambda_1 > 0$ मौजूद है जैसे कि (P_λ^1) कम से कम दो समाधान स्वीकार करता है जब $0<\lambda< \Lambda_1$ हो। हमने पिछले अनुभाग में अलग-अलग $q \geq 1$, $\alpha = 2^*_s - 1$ केस का भी अध्ययन किया है। यहां, हम λ की उपयुक्त सीमा के लिए $L^1_{\{loc\}}(\Omega)$ में कमजोर समाधानों की बहुतायत दिखाने के लिए गैर चिकनी विश्लेषण से परिणामों का उपयोग करते हैं। हम इस समस्या के लिए प्राप्त कमजोर समाधानों की नियमितता भी दिखाते हैं, जो कि $L^\infty(\Omega) \cap C^\alpha_{\{loc\}}(\Omega)$ से संबंधित हैं जहां α (0,1) है ।

तीसरे अध्याय में निम्नलिखित एकवचन पैराबॉलिक समीकरण के लिए अस्तित्व और स्थिरीकरण परिणाम शामिल हैं जहा फ्रॅक्शनल लपलाशन शामिल हैं

(P_t^s): $u_t + (-\Delta)^s u = u^{-q} + f(x,u)$, $u > 0$ Λ_T में, $u = 0$ Γ_T में, $u(0,x) = u_0(x)$ R^n में,

जहां $\Lambda_T = (0,T) \times \Omega$, $\Gamma_T = (0,T) \times R^n \backslash \Omega$, Ω में एक बाध्य डोमेन है R^n चिकनी सीमा $\partial\Omega$ (कम से कम C^2), $n > 2s$, $0<s<1$, $q > 0$, जैसे कि $q(2s-1)<(2s+1)$ और $T > 0$ फ़ंक्शन $(x, y)\ \Omega \times R \rightarrow f(x, y)$ में फ़ंक्शन को नीचे से बाध्य किया जाता है, स्थानीय रूप से लिपस्कित्ज़ दूसरा चर और समान रूप से x Ω में और यह संतुष्ट है

$$\lim_{y \to \infty} sup \frac{f(x,y)}{y} < \lambda_{1,s}$$

जहां $\lambda_{1,s}$ $R^n \backslash \Omega$ में डिरिच्लेत सीमा स्थिति के साथ $(-\Delta)^s$ का पहला आईगेनवलुए है। हम u_0 को उचित शंकु की स्थिति को पूरा करने पर (P_t^s) के कमजोर समाधान के अस्तित्व और विशिष्टता को साबित करते हैं। हम अंतर्निहित यूलर विधि के साथ समय में अर्ध-विघटन का उपयोग करते हैं और हमारे परिणामों को साबित करने के लिए स्थिर समस्या का अध्ययन करते हैं। जब हम अपने प्रारंभिक कार्य u_0 को नियमित करते हैं तो हम (P_t^s)के समाधान पर अतिरिक्त नियमितता भी दिखाते हैं।

हमने एक गैर-एकल एकवचन समस्या के लिए चौथे अध्याय में तीन समाधान प्रमेय साबित कर दिया है। निश्चित रूप से हम निम्नलिखित नोंलोकाल एकवचन समस्या माना जाता है

(P_λ^2): $(-\Delta)^s u = \lambda\frac{f(u)}{u^q}$, $u > 0$ Ω में, $u = 0$ $R^n \setminus \Omega$ में,

जहां $0<s<1$, $0<q<1$, $\lambda > 0$ और Ω R^n में एक चिकनी बाध्य डोमेन है। यहां हम $f: [0,\infty) \to [0,\infty)$ को एक निरंतर नांदेसरीसिंग मानचित्र होने के लिए $\lim_{u\to\infty}\frac{f(u)}{u^{\{q+!\}}} = 0$ । f पर कुछ अतिरिक्त मान्यताओं के तहत, हम साबित करते हैं कि (P_λ^2) में λ की एक निश्चित सीमा में कम से कम तीन विशिष्ट समाधान हैं। हम अपने परिणामों को साबित करने के लिए उप और सुपरसोल्यूशन की विधि और अमान [8] द्वारा एक महत्वपूर्ण बिंदु प्रमेय का उपयोग करते हैं। इसके अलावा हम दिखाते हैं कि जब λ काफी बड़ा हो जाता है, (P_λ^2) का एक अनूठा समाधान होता है।

पांचवें अध्याय में, हमने एक दोगुनी नॉनोकोकल ब्रेज़िस-निरेनबर्ग प्रकार चोवार्ड समीकरण का अध्ययन किया

(P_λ^3): $(-\Delta)^s u = \lambda u + \left(\int_\Omega \frac{|u(y)|^{2^*_{\{\mu,s\}}}}{|x-y|^\mu}dy\right)|u|^{2^*_{\{\mu,s\}}-2}u$ Ω में, $u = 0$ $R^n \setminus \Omega$ में,

जहां Ω R^n में लिप्सचिट्ज़ सीमा के साथ एक बाध्य डोमेन है, λ R^n में, $0<s<1$, $2^*_{\{\mu,s\}} = \frac{2n-\mu}{n-2s}$ हार्डी-लिटिलवुड-सोबोलिव असमानता, $0<\mu<n$ और $n> 2s$ के अर्थ में ऊपरी महत्वपूर्ण एक्सपोनेंट है। माउंटेन पास प्रमेय और लिंकिंग प्रमेय का उपयोग करके हमने पैरामीटर $ \ la $ की सीमा के आधार पर (P_λ^3) कमजोर समाधान का अस्तित्व साबित कर दिया। हम यह भी दिखाते हैं कि प्रत्येक समाधान कुछ $\alpha>0$ Ω और s के आधार पर $L^\infty(\Omega)\cap C^\alpha(\overline{\Omega})$ में है संतुष्ट $\alpha< \min\{1,1-s\}$। इसके अलावा पोहोज़ेव पहचान का उपयोग करते हुए, हम साबित करते हैं कि जब $\lambda<0$ और Ω मूल के संबंध में सख्ती से स्टार आकार दिया जाता है तो (P_λ^3) कोई सकारात्मक नॉनट्रिविअल समाधान स्वीकार करता है।

इस आखिरी अध्याय में, हमने समीकरणों की एक प्रणाली को माना जो आंशिक लैपलियान और महत्वपूर्ण चक्करदार गैर-रेखाचित्र शामिल है

$(P_{\{\lambda,\delta\}})$: $(-\Delta)^s u = \lambda|u|^{q-2}u + \int_\Omega \frac{|v(y)|^{2^*_{\{\mu,s\}}}}{|x-y|^\mu}dy\,|u|^{2^*_{\{\mu,s\}}-2}u$ Ω में

$(-\Delta)^s v = \delta|v|^{q-2}v + \int_\Omega \frac{|u(y)|^{2^*_{\{\mu,s\}}}}{|x-y|^\mu}dy\,|v|^{2^*_{\{\mu,s\}}-2}v$ Ω में

$u = v = 0$ $R^n \setminus \Omega$ में,

जहां $n> 2s$, $0<s<1$, $0<\mu<n$, $1<q<2$, Ω R^n में एक चिकनी बाध्य डोमेन है और $\lambda, \delta > 0$ असली पैरामीटर हैं। नेहारी कई गुना के सबसेट पर कम से कम उपयोग करके, हम साबित करते हैं कि सिस्टम $(P_{\{\lambda,\delta\}})$ कम से कम दो नॉनट्रिविअल समाधान स्वीकार करता है जब भी $0 < \lambda^{\left\{\frac{2}{2-q}\right\}} + \delta^{\left\{\frac{2}{2-q}\right\}} < \Theta$ अगर $\mu\leq 4s$ और $0 < \lambda^{\left\{\frac{2}{2-q}\right\}} + \delta^{\left\{\frac{2}{2-q}\right\}} < \Theta_0$ अगर $\mu> 4s$ कुछ $\Theta_0, \Theta > 0$ के लिए जहां संभवतः $\Theta_0 \leq \Theta$।

Contents

List of Symbols

Symbol	**Meaning**
$\mathscr{C}A$	Complement of the set $A \subset \mathbb{R}^n$ that is $\mathbb{R}^n \setminus A$.
$\lvert A \rvert$	Lebesgue measure of a set $A \subset \mathbb{R}^n$.
$K \Subset D$	K is compact subset of D.
1_A	Characteristic function of the set A.
$B_r(x)$	Ball of radius r centered at x in $\mathbb{R}^n$.
B_r	Ball of radius r centered at the origin in $\mathbb{R}^n$.
$C_c^\infty(\Omega)$	Set of infinitely differentiable functions with compact support in Ω.
$\overline{A}^B$	Closure of A in B.
$\lvert f \rvert_p$	Norm of f in $L^p(\Omega)$, $1 \leq p < \infty$.
W^*	Dual of any Banach space W.
$\lVert \cdot \rVert_*$	Norm in dual space W^*.
$H^s(\Omega)$	Sobolev space of order s and exponent 2
$(-\Delta)_p^s$	p-fractional Laplacian, $1 < p < \infty$.
$\lVert u \rVert_\infty$	Norm of u in $L^\infty(\Omega)$ space.
$\lVert u \rVert_{X_0^p}$	Norm of u in the space X_0^p.
$\lVert u \rVert$	Norm of u in the space X_0.
$L_{loc}^p(\Omega)$	Locally p-integrable function space.