SOME FINDINGS IN RADO NUMBERS

SRASHTI DWIVEDI

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2020

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by

SRASHTI DWIVEDI

Department of Mathematics

Submitted

in fulfilment of the requirements of the degree of Doctor of Philosophy

to the

INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2020

Certificate

This is to certify that the thesis entitled **Some findings in Rado numbers** submitted by **Ms. Srashti Dwivedi** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
July 2020

Dr. Amitabha Tripathi
Professor
Department of Mathematics
Indian Institute of Technology Delhi
New Delhi - 110 016

Acknowledgements

First and foremost I would like to thank my supervisor, Dr. Amitabha Tripathi, whose guidance and support was indispensable. I learned a lot while working with him. His dedication towards his work, teaching and Mathematics in general was inspiring in many ways. The way he approaches any Mathematical problem and his persistence to look for a concise and simple proof is what I would like to take as a lesson with me.

I would like to thank all the faculty members, administrative staff or staff in general and students at IIT Delhi, for helping me grow as a student and a researcher. I would also like to thank members of my SRC for their time and insightful remarks that helped me understand where I lacked or what more I could do to improve things further.

This would be incomplete if I don't thank my friends who supported and cheered me throughout this journey. Mayank Gupta, Pooja Punyani, Puja Bansal, Rabiya Kamra, Anupam Gautam, Anjalika, Akansha, Divya di, Kavita, Annu, Sarab, everyone in Room no.168, Udit, Ishan, Sidhesh, Yash, Edgar, Archana, Surbhi, Moushmi Ma'am and to those whom I might have missed here, thanks again.

Lastly, I would thank my family for their love and encouragement. Mom, Dad, Bua, Anu di and Guddu di I can't even imagine this particular journey or my life without your support, motivation, occasional scoldings and love. Mom you pushed me when I wanted to give up and Dad your hard work and fighting spirit served as an example all these years for us not to shy away from it ever.

Srashti Dwivedi

July 2020

Abstract

Let $\mathcal{E}$ be a single or a system of equations and let r be a positive integer, $r > 1$. The Rado number, if it exists, is the least positive integer R such that every r-colouring of integers from the interval $[1, R]$ admits a monochromatic solution to $\mathcal{E}$.

We mostly consider the case $r = 2$, and for a variety of classes of equation or equations (i) consider the problem of existence of the Rado number, and (ii) determine upper and lower bounds, and in some cases, exact values of Rado numbers. We also consider Disjunctive Rado numbers. Disjunctive Rado number for a system of two equations $\mathcal{E}_1, \mathcal{E}_2$ is the least positive integer D such that every r-colouring of the integers from the intervals $[1, D]$ contains a monochromatic solution to either $\mathcal{E}_1$ or $\mathcal{E}_2$.

We cover Rado numbers, Disjunctive Rado numbers, Schaal equation and its generalization in this thesis. The equations that we worked with are

$$\sum_{i=1}^{m-1} a_i x_i - x_m = c,$$

$$\sum_{i=1}^{m-2} x_i + a x_{m-1} - x_m = c,$$

$$\sum_{i=1}^{m-2} a x_i + b x_{m-1} = b x_m,$$

and

$$a(x_1 + x_2 + x_3) = 3x_4.$$

Our main focus is non-homogeneous equations and how the coefficient set impacts the existence of Rado number and Disjunctive Rado number, along with the constant involved and number of variables.

सार

मानते हैं कि $\mathcal{E}$ एक समीकरण या समीकरणों की एक प्रणाली है और r एक सकारात्मक पूर्णांक है, $r > 1$। यदि राडो संख्या का अस्तित्व है, तो वह इस प्रकार का न्यूनतम सकारात्मक पूर्णांक R है कि पूर्णांकों के $[1, R]$ अन्तराल में प्रत्येक r-रंजन $\mathcal{E}$ के लिए एक एक रंग के समाधान को स्वीकार करता है।

हम ज़्यादातर $r = 2$ स्थिति पर ध्यान देते हैं, और समीकरणों या समीकरणों के विभिन्न वर्गों के लिए (i) राडो संख्या के अस्तित्व की समस्या पर विचार करते हैं, और (ii) राडो संख्याओं की ऊपरी और निचली सीमाऐं, और कुछ स्तिथियों में उसका सटीक मूल्य, निर्धारित करते हैं। हम वियोगी राडो संख्याओं (Disjunctive Rado Numbers) पर भी विचार करते हैं। दो समीकरणों $\mathcal{E}_1$, $\mathcal{E}_2$ की प्रणाली के लिए वियोगी राडो संख्या एक ऐसा सकारात्मक पूर्णांक D है जिससे $[1, D]$ अन्तराल में पूर्णांकों के हर r-रंजन में $\mathcal{E}_1$ या $\mathcal{E}_2$ के लिए एक एक रंग का समाधान मौजूद है।

हम इस शोध प्रबंध में राडो संख्याओं, वियोगी राडो संख्याओं, शाल (Schaal) समीकरण और उसके सामान्यकरण पर चर्चा करेंगे। जिन समीकरणों के साथ हम काम कर रहे हैं वे हैं:

$$\sum_{i=1}^{m-1} \mathrm{a_i} x_i - x_m = c,$$
$$\sum_{i=1}^{m-2} x_i + a x_{m-1} - x_m = c,$$
$$\sum_{i=1}^{m-2} a x_i + b x_{m-1} = b x_m,$$

और

$$a(x_1 + x_2 + x_3) = 3x_4.$$

हमारा प्रमुख ध्यान गैर-समरूप समीकरणों पर है और इस पर कि गुणक समूह राडो संख्या और वियोगी राडो संख्या के अस्तित्व पर, विश्लेषण में शामिल स्थिर संख्या पर और चार संख्याओं की गिनती पर प्रभाव किस प्रकार से डालता है।

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List of Symbols

Symbol	**Meaning**
WLOG	Without loss of generality
Valid colouring	Colouring that avoids a monochromatic solution
Monochromatic solution	Solution in one colour only
$\gcd(a_1, \ldots, a_i)$	gcd of $\{a_1, \ldots, a_i\}$
$\text{lcm}(a_1, \ldots, a_i)$	lcm of $\{a_1, \ldots, a_i\}$
$\mathbb{R}$	Set of Real numbers
mod(n)	Set of non-negative integers {0, 1, . . . , n-1}
$\mathbb{Z}$	Set of Integers
$\mathbb{Z}^+$	Set of positive integers
$\mathbb{N}$	Set of Natural numbers
$[n]^k$	$\{X : X \subseteq \{1, \ldots, n\} \text{ and } \lvert X\rvert = k\}$
$[n]^{\leq k}$	$\{X : X \subseteq \{1, \ldots, n\} \text{ and } \lvert X\rvert \leq k\}$