EXPERIMENTAL STUDIES ON THE NONLINEAR VIBRATIONS OF BEAMS AND CYLINDRICAL SHELLS

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EXPERIMENTAL STUDIES ON THE NONLINEAR VIBRATIONS OF BEAMS AND CYLINDRICAL SHELLS

by

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in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

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Certificate

This is to certify that the thesis entitled **"Experimental Studies on the Nonlinear Vibrations of Beams and Cylindrical Shells"** being submitted by **Mr. Ajay Kumar** to the Indian Institute of Technology Delhi for the award of degree of **Doctor of Philosophy** in Applied Mechanics is a record of original, bonafide research work carried out by him under my supervision and guidance. The thesis work, in my opinion, has reached the requisite standard fulfilling the requirements for the degree of Doctor of Philosophy.

The results contained in this thesis have not been submitted in part or in full, to any other university or institute for the award of any degree or diploma.

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(Ajay Kumar)

Abstract

The thin walled straight/curved beams and circular/noncircular cylindrical shells undergoing vibrations with amplitude of the order of their thickness or greater depict a significant effect of geometric nonlinearity involving traveling wave response, external resonances at forcing frequencies away from linear free vibration frequencies, internal resonance, participation of sub-or super-harmonics, quasi periodic and chaotic responses among the other nonlinear vibration characteristics. The experimental investigations on the nonlinear vibration of structures are important to validate their numerical prediction and to explore new phenomena/modal interactions.

The thesis deals with the nonlinear vibration of inextensional cantilever beams with accurate representation of curvature/inertia nonlinearities/spatial discretization, response of curved beams under symmetric/anti-symmetric excitation, experimental study of large amplitude vibrations of cantilever circular cylindrical shells and investigation of travelling wave response, and the study on the nonlinear response of elliptical cylindrical shell under harmonic excitation through experiments.

An experimental set-up involving non-contact harmonic excitation of beams/shells, acceleration measurement and data processing system is employed for the nonlinear vibration study of cantilever, curved beams, circular and noncircular cylindrical shells made up of aluminum. The nature of traveling wave response in cylindrical shells is investigated using a series of accelerometers along the circumference/high speed cameras. The periodic response of inextensible straight cantilever beams and curved beams is also investigated numerically using finite element and shooting technique coupled with the Newmark’s direct time integration and arc length continuation.

The nonlinear response of cantilever beams depicts mild hardening for the first mode and the increasing degree of softening nonlinearity for second/higher modes. The third order approximation of nonlinear inertia/curvature terms leads to smaller hardening and softening for first and second/third modes, respectively. The single mode approximation of transverse displacement leads to significantly greater degree of hardening for first mode, smaller degree of softening for second and greater degree of softening nonlinearity for fourth mode compared to more accurate solution. The coupled mode response involving symmetric and anti-symmetric modes is observed for the direct excitation of both symmetric and anti-symmetric modes of fixed-fixed curved beams.

The cantilever circular cylindrical shell depicts increasing degree of softening nonlinearity for excitation in the neighbourhood of modes with increasing number of circumferential waves, and the travelling wave response is observed for modes with circumferential wave number equal to 3 and 5.

The elliptical cylindrical shell depicts softening nonlinearity, travelling waves over part of the circumference or along the full circumference with/without variable amplitude depending upon the forcing frequency. The participation of different modes involving 1:1, 2:1, 3:1 external, 1:1, 1:2, 2:3 internal resonances and the presence of harmonics including multiples of 1/2, 1/3 are reported. For the excitation of the elliptical cylindrical shell between semi minor and major axes, the response is found to be predominantly travelling wave type for a greater range of excitation frequency.

सारांश

अपनी मोटाई के प्रमाण आयाम से कंपन कर रही पतली सरल / वक्राकार पट्टियाँ एवं बेलनाकार खोल, असमानुपातिक विकृति-विस्थापन संबंध का महत्वपूर्ण प्रभाव जैसे

परिधीय दिशा मे प्रगामी तरंग व्यवहार, अरेखीय आज़ाद कंपन की आवृत्तियों से भिन्न आवृत्ति वाले बाह्य बलों के अधीन अनुनाद, आंतरिक अनुनाद, भिन्नांक/ पूर्णांक हारमोनिक्स की सहभागिता, लगभग पुनरावृत्त एवं अव्यवस्थित कंपन आदि अरेखीय कंपन विशेषताओं को दर्शाते हैं| संरचनाओ के अरेखीय कंपन का प्रायोगिक अध्ययन, अभिकलनात्मक अनुमानों के प्रमाणन एवं नये प्रकार के व्यवहार, विभिन्न आज़ाद कंपन रूपो के पारस्परिक प्रभाव की खोज आदि करने की दृष्टि से महत्त्वपूर्ण है|

प्रस्तुत शोध प्रबंध, अखिंचनीय केंद्रीय अक्ष वाली कैन्टीलीवर पट्टी का अरेखीय कंपन, अरेखीय वक्रता / जड़त्व के विशुद्ध प्रतिरूप / स्पेशियल अंतर्वेशन के साथ, मध्य बिन्दु के सापेक्ष सममित / असममित बलों के अधीन वक्राकार पट्टियों का कंपन व्यवहार, कैन्टीलीवर वृत्तीय बेलनाकार कोश के बृहद् आयामीय कम्पनों का प्रायोगिक अध्ययन एवं प्रगामी तरंग व्यवहार का अन्वेषण तथा दीर्घवृत्तीय बेलनाकार कोश का सन्नादी बाह्य बलों के अधीन अरेखीय कंपन व्यवहार का प्रायोगिक अध्ययन प्रस्तुत करता है|

एल्युमीनियम से निर्मित कैन्टीलीवर, वक्राकार पट्टी, वृत्तीय एवं दीर्घवृत्तीय बेलनाकार कोशों के अरेखीय कंपन व्यवहार के अध्ययन के लिये अस्पर्शी आवर्ती उत्तेजन, त्वरण मापन एवं प्रक्रियण प्रणालियो का उपयोग किया गया है| बेलनाकार कोशों मे प्रगामी तरंग व्यवहार का अध्ययन, परिधीय दिशा मे त्वरणमापियो की एक शृंखला एवं द्रुतगतीय प्रतिबिम्बमापी के द्वारा किया गया है| अखिंचनीय केंद्रीय अक्ष वाली कैन्टीलीवर

एवं वक्राकार पट्टियों के आवर्ती कंपन व्यवहार का अध्ययन अल्पांश, शूटिंग तकनीक, न्यूमार्क की अंकगणितीय समय समाकलन एवं बहुआयामी चाप लम्बाई सांतत्य विधियो के प्रयोग से अन्वेषित किया गया है|

कैन्टीलीवर का अरेखीय कंपन व्यवहार, प्रथम विधा मे सूक्ष्म कठोर प्रकृति एवं द्वतीय/ उच्च विधाओ मे बढ़ती नरम प्रकृति दर्शाता है| अरेखीय जड़त्व / वक्रता का त्रिघातीय सन्निकटन प्रथम तथा द्वतीय / तृतीय विधाओ मे क्रमश: कठोर एवं नरम व्यवहार मे कमी दर्शाता है| अनुप्रस्थ विस्थापन का एकल विधा सन्निकटन परिशुद्ध हल की तुलना में, प्रथम विधा मे कठोरता मे प्रचुर वृद्धि, द्वितीय विधा मे नरमता मे कमी एवं चतुर्थ विधा मे नरमता मे वृद्धि दर्शाता है| मध्य बिन्दु के सापेक्ष सममित एवं असममित विधाओ का समिश्रित व्यवहार, वक्राकार पट्टियों के सममित एवं असममित विधाओ के उत्तेजन मे पाया गया है|

कैन्टीलीवर वृत्तीय बेलनाकार कोश, परिधीय तरंग क्रम बढ़ने के साथ बढ़ती हुयी अरेखीय नरम व्यवहार तथा परिधीय तरंग क्रम 3 और 5 के लिए प्रगामी तरंग व्यवहार दर्शाती है|

दीर्घवृत्तीय बेलनाकार कोश, नरम अरेखीय व्यवहार, बाह्य उत्तेजक बल की आवृत्ति पर निर्भर परिधीय अल्पांश/ पूर्ण परिधि मे प्रगामी तरंगें स्थिर / बदलते आयाम के साथ दर्शाती है| विभिन्न विधाओ की सहभागिता 1:1, 2:1, 3:1 बाह्य एवं 1:1, 1:2,

2:3 आंतरिक अनुनादो के कारण तथा विभिन्न (पुर्णाक़ (1,2,3,...) / भिन्नांक (1/2, 1/3 आदि के गुणज) हारमोनिक्स का अरेखीय कंपन व्यवहार मे उपस्थिति दर्शायी गयी है|

दीर्घ एवं लघु अक्षों के बीच में उत्तेजित दीर्घवृत्तीय बेलनाकार कोश में मुख्यतया प्रगामी तरंग व्यवहार बड़े उत्तेजन आवृत्ति परिसर में पाया गया है|

Table of contents

Certificate *i*

Acknowledgements *ii*

Abstract *iii*

Table of Contents *v*

List of Figures *viii*

List of Tables *xvi*

Nomenclature *xvii*

Chapter 1 Introduction and Literature Review 1

1.1 Introduction 1

1.2 Literature review 2

1.2.1 Cantilever beams 2

1.2.2 Curved beams 5

1.2.3 Circular cylindrical shells 7

1.2.4 Noncircular cylindrical shells 10

1.3 Motivation and Objectives 11

1.4 Organization of the Thesis 13

Chapter 2 Nonlinear Vibrations of Cantilever Beam 15

2.1 Experimental Setup 15

2.2 Formulation and Solution Methodology 17

2.2.1 Equation of Motion 17

2.2.2 Finite Element Formulation 19

2.2.3 Analytical Solution 21

2.3 Solution Methodology 22

2.4 Results and Discussion 26

2.5 Concluding Remarks 43

Chapter 3 Nonlinear Vibrations of Curved Beams 44

3.1 Experimental Setup 44

3.2 Finite Element Modelling 46

3.3 Results and Discussion 47

3.3.1 Symmetric Excitation of curved beam (span angle = 80^0) 48

3.3.2 Anti-Symmetric Excitation of curved beam (span angle = 80^0) 51

3.3.3 Symmetric Excitation of curved beam (span angle = 130^0) 55

3.3.4 Anti-Symmetric Excitation of curved beam (span angle = 130^0) 58

3.4 Concluding Remarks 60

Chapter 4 Nonlinear Vibrations of Cantilever Cylindrical Shell 62

4.1 Experimental Setup 62

4.2 Results and Discussion 66

4.3 Concluding Remarks 86

Chapter 5 Nonlinear Vibrations of Elliptical Cylindrical Shell 88

5.1 Experimental Setup 88

5.2 Finite Element Modelling 89

5.3 Results and Discussion 90

5.4 Concluding Remarks 127

Chapter 6 Conclusions and Scope for Future Research Work 128

6.1 Conclusions from Present Work 128

6.2 Scope for Future Research Work 131

References 132

Appendices 140

List of publication from the thesis 143

Brief biodata of the author 144

List of Figures

Figure No.	Title	Page No.
2.1	Schematic diagram of experimental set up for cantilever beam.	15
2.2	Experimental set up for cantilever beam.	16
2.3	Cantilever beam and coordinate system.	18
2.4	Finite element discretization of the cantilever beam and local coordinate of each element.	20
2.5	Acceleration versus forcing frequency curves of cantilever beam for mode 1 with excitation amplitudes of 2 and 3 Volts.	26
2.6	Acceleration versus forcing frequency curves of cantilever beam for mode 2 with excitation amplitudes of 2 and 3Volts.	27
2.7	Transverse displacement amplitude at $x = L$ versus forcing frequency curves of cantilever beam in the neighbourhood of mode 1 (ξ_1=0.01).	28
2.8	Response history and frequency spectrum of cantilever beam corresponding to point 1 (Figure 2.7, ω_F/ω_{n1} =1.0185).	29
2.9	Transverse displacement amplitude at $x = L$ versus forcing frequency curves of cantilever beam in the neighbourhood of mode 2 (ξ_2=0.01).	30
2.10	Response history and frequency spectrum of cantilever beam corresponding to point 1 (Figure 2.9, ω_F/ω_{n2} =0.7812).	31
2.11	Response history and frequency spectrum of cantilever beam corresponding to point 2 (Figure 2.9, ω_F/ω_{n2} =0.91).	31

2.12 Response history and frequency spectrum of cantilever beam corresponding to point 3 (Figure 2.9, ω_F/ω_{n2} =0.8972). 32

2.13 Response history and frequency spectrum of cantilever beam corresponding to point 4 (Figure 2.9, ω_F/ω_{n2} =0.9293). 32

2.14 Transverse displacement amplitude at $x = L$ versus forcing frequency curves of cantilever beam in the neighbourhood of mode 3 (ξ_3=0.01). 33

2.15 Response history and frequency spectrum of cantilever beam corresponding to point 1 (Figure 2.14, ω_F/ω_{n3} =0.64937). 34

2.16 Response history and frequency spectrum of cantilever beam corresponding to point 2 (Figure 2.14, ω_F/ω_{n3} =0.795). 35

2.17 Response history and frequency spectrum of cantilever beam corresponding to point 3 (Figure 2.14, ω_F/ω_{n3} =0.7331). 36

2.18 Response history and frequency spectrum of cantilever beam corresponding to point 4 (Figure 2.14, ω_F/ω_{n3} =0.9050). 37

2.19 Transverse displacement amplitude versus forcing frequency curves of cantilever beam in the neighbourhood of mode 4. 38

2.20 Response history and frequency spectrum of cantilever beam corresponding to point 1 (Figure 2.19, ω_F/ω_{n4} =0.7841). 39

2.21 Response history and frequency spectrum of cantilever beam corresponding to point 2 (Figure 2.19, ω_F/ω_{n4} =0.8095). 40

2.22 Response history and frequency spectrum of cantilever beam corresponding to point 3 (Figure 2.19, ω_F/ω_{n4} =0.9219). 41

2.23 Response history and frequency spectrum of cantilever beam corresponding to point 4 (Figure 2.19, ω_F/ω_{n4} =0.899456). 42

3.1 Experimental Setup. 45

3.2 Photograph of curved beam with exciter. 45

3.3 Curved beam geometry. 46

3.4 Acceleration versus forcing frequency curves of curved beam (span angle = 80^0) excited at location B with excitation amplitudes of 1.2, 2.4 and 3.5 Volts: (a) Response at B, (b) Response at A. 49

3.5 Steady state response history and frequency spectra of curved beam of span angle 80^0 excited at location B (Excitation amplitude = 3.5 V). 50

3.6 Acceleration versus forcing frequency curves of curved beam (span angle = 80^0) excited at location A with excitation amplitudes of 1, 2 and 3 Volts: (a) Response at B, (b) Response at A. 53

3.7 Steady state response history and frequency spectra of curved beam of span angle 80^0 excited at location B (Excitation amplitude = 3.0V). 54

3.8 Variation of radial displacement along length at different time instants of a cycle. (Excitation amplitude = 3.0V). 55

3.9 Acceleration versus forcing frequency curves of curved beam (span angle = 130^0) excited at location B with excitation amplitudes of 0.5, 1.0 and 1.5 volts: (a) Response at B, (b) Response at A. 56

3.10 Steady state response history and frequency spectrum of curved beam of span angle 130^0 excited at location B (excitation amplitude = 1.5V). 57

3.11 Acceleration versus forcing frequency curves of curved beam (span angle = 130^o) excited at location A with excitation 59 amplitudes of 0.2, 0.4 and 0.6 Volts: (a) Response at B, (b) Response at A.

3.12 Steady state response history and frequency spectrum of curved beam of span angle 130^0 excited at location A (Excitation amplitude = 0.6V). 60

4.1 Experimental setup. 63

4.2 Location of accelerometers on the circumference of shell. 63

4.3 Digital image correlation experimental setup. 65

4.4 3D speckle image with high speed camera. 65

4.5 2D speckle image with high speed camera. 65

4.6 Comparison of acceleration at top versus forcing frequency curve of cantilever shell with ANSYS results. 68

4.7 Comparison of displacement at point A versus time curve of cantilever shell obtained from accelerometer data and Vic-3D for forcing frequency: (a) 29.4 Hz (b) 29.8 Hz. 69

4.8 Steady state response history and frequency spectra of shell at 28 Hz forcing frequency (Excitation voltage amplitude = 6V). 72

4.9 Circumferential variation of radial acceleration corresponding to different points marked in Figure 4.8A (ω_F = 28 Hz, excitation amplitude = 6 Volts). 73

4.10 Steady state response history, frequency spectra and phase angle plots of shell at 29.2 Hz forcing frequency (Excitation voltage amplitude = 6V). 73

4.11 Circumferential variation of radial acceleration corresponding to different points marked in Figure 4.10A (ω_F = 29.2 Hz, excitation amplitude = 6 Volts). 74

4.12 Steady state response history, frequency spectra and phase angle plots of shell at 29.6 Hz forcing frequency (Excitation voltage amplitude = 6V). 75

4.13 Circumferential variation of radial acceleration corresponding to different points marked in Figure 4.12A (ω_F = 29.6 Hz, excitation amplitude = 6 Volts). 76

4.14 Steady state response history, frequency spectra and phase angle plots of shell at 30 Hz forcing frequency (Excitation voltage amplitude = 6V). 76

4.15 Circumferential variation of radial acceleration corresponding to different points marked in Figure 4.14A (ω_F = 30 Hz, excitation amplitude = 6 Volts). 77

4.16 Steady state response history, frequency spectra and phase angle plots of shell at 29.6 Hz forcing frequency (Excitation voltage amplitude = 12V). 78

4.17 Circumferential variation of radial acceleration corresponding to different points marked in Figure 4.16A (ω_F = 29.6 Hz, excitation amplitude = 12 Volts). 79

4.18 Deformed shapes of circular cylindrical shell for 29 Hz forcing frequency at equi-spaced time instants in one cycle. 80

4.19 Deformed shapes of circular cylindrical shell for 29.6 Hz forcing frequency at equi-spaced time instants in one cycle. 81

4.20 Acceleration at locations A and G versus forcing frequency of shell excited in the neighbourhood of mode with n =3. 82

4.21 Acceleration versus forcing frequency at location A of shell excited in the neighbourhood of mode with n = 4. 82

4.22 Acceleration versus forcing frequency at location A of shell excited in mode with n = 5. 83

4.23 Acceleration versus forcing frequency at location G of shell excited in mode with $n = 5$. 83

4.24 Steady state response history, frequency spectra and phase angle plots of shell at 89.8 Hz forcing frequency (Excitation voltage amplitude =12V). 85

4.25 Circumferential variation of radial acceleration corresponding to different points marked in Figure 4.24A (ω_F = 89.8 Hz, excitation amplitude = 12 Volts). 86

5.1 Experimental setup for elliptical cylindrical shell. 88

5.2 Location of excitation points and accelerometers. 89

5.3 Frequency response spectra of elliptical cylindrical shell at location A_1 due to impact hammer excitation. 90

5.4 Free vibration mode shapes of the elliptical cylindrical shell. 92

5.5 Frequency response of elliptical cylindrical shell at A_1 due to excitation at E_1. 96

5.6 Response of elliptical cylindrical shell at ω_F = 81 Hz (Point 1, Figure 5.5, excitation amplitude = 7.5 Volts). 97

5.7 Response of elliptical cylindrical shell at ω_F = 83.5 Hz (Point 2, Figure 5.5, excitation amplitude = 7.5 Volts). 99

5.8 Response of elliptical cylindrical shell at ω_F = 85 Hz (Point 3, Figure 5.5, excitation amplitude = 7.5 Volts). 100

5.9 Response of elliptical cylindrical shell at ω_F = 87 Hz (Point 4, Figure 5.5, excitation amplitude = 7.5 Volts). 101

5.10 Response of elliptical cylindrical shell at ω_F = 89 Hz (Point 5, Figure 5.5, excitation amplitude = 7.5 Volts). 102

5.11 Response of elliptical cylindrical shell at ω_F = 91 Hz (Point 6, Figure 5.5, excitation amplitude = 7.5 Volts). 103

5.12 Response of elliptical cylindrical shell at ω_F = 97.5 Hz (Point 7, Figure 5.5, excitation amplitude = 7.5 Volts). 104

5.13 Response of elliptical cylindrical shell at ω_F = 99.6 Hz (Point 8, Figure 5.5, excitation amplitude = 7.5 Volts). 106

5.14 Response of elliptical cylindrical shell at ω_F = 101.5 Hz (Point 9, Figure 5.5, excitation amplitude = 7.5 Volts). 107

5.15 Response of elliptical cylindrical shell at ω_F = 123.2 Hz (Point 10, Figure 5.5, excitation amplitude = 7.5 Volts). 108

5.16 Response of elliptical cylindrical shell at ω_F = 149 Hz (Point 11, Figure 5.5, excitation amplitude = 7.5 Volts). 109

5.17 Response of elliptical cylindrical shell at ω_F = 152 Hz (Point 12, Figure 5.5, excitation amplitude = 7.5 Volts). 111

5.18 Response of elliptical cylindrical shell at ω_F = 153.3 Hz (Point 13, Figure 5.5, excitation amplitude = 7.5 Volts). 113

5.19 Response of elliptical cylindrical shell at ω_F = 180 Hz (Point 14, Figure 5.5, excitation amplitude = 7.5 Volts). 115

5.20 Response of elliptical cylindrical shell at ω_F = 193.2 Hz (Point 15, Figure 5.5, excitation amplitude = 7.5 Volts). 116

5.21 Response of elliptical cylindrical shell at ω_F = 200 Hz (Point 16, Figure 5.5, excitation amplitude = 7.5 Volts). 118

5.22 Response of elliptical cylindrical shell at ω_F = 203.5 Hz (Point 17, Figure 5.5, excitation amplitude = 7.5 Volts). 119

5.23 Frequency response of elliptical cylindrical shell at E_1 due to excitation at E_3, E_4, E_5. 122

5.24 Steady state response history and frequency spectra of elliptical shell at E_1 (Excitation amplitude 7.5V at E_3). 123

5.25 Variation of maximum acceleration (at x/L=0.5) along circumference of the elliptical cylindrical shell (Excitation amplitude = 7.5 Volts at E_3). 126

5.26 Variation of maximum acceleration (at y/C = 0.02) along the length of elliptical cylindrical shell (Excitation amplitude = 7.5 Volts at E_3). 126

List of Tables

Table No.	Title	Page No
2.1	Dimensions and material properties of cantilever beam.	17
3.1	First few linear free vibration frequencies of the curved beams.	48
4.1	Dimensions and material properties of the cylindrical shell.	64
4.2	Radius (R) of the manufactured shell at free end versus angular position ($\tilde{\theta}$) measured from excitation point in anti-clockwise direction.	64
4.3	Natural frequencies of the cylindrical shell (number of longitudinal quarter waves = 1).	67
5.1	Comparison of natural frequencies of the elliptical cylindrical shell.	91

Nomenclature

English Notations

A : Area

$\mathrm{A}_i \ \ (i = 0,1,2,...)$: Harmonic amplitude

b : Width of beams

C : Circumferential length of shells

$\mathbf{d}, \dot{\mathbf{d}}, \ddot{\mathbf{d}}$: Displacement, velocity and acceleration vectors

$\mathbf{d}_\mathbf{i}$: Vector of elemental degrees of freedom of i^{th} element

$\mathbf{D}$: Global damping matrix

$\tilde{\mathbf{D}}$: Modal damping matrix

$\mathbf{D}^i$: Elemental damping matrix

E : Young's modulus

$\mathbf{F}$: Global load vector

$\mathbf{F}^i$: Elemental load vector

$\tilde{\mathbf{F}}$: Modal load vector

h : Mean thickness

$\mathbf{H}$: Vector of Hermite interpolation functions

$\mathbf{H}_{,i}$, $\mathbf{H}_{,j}$: $\dfrac{\partial \mathbf{H}}{\partial x_i}, \dfrac{\partial \mathbf{H}}{\partial x_j}$

$\mathbf{H}_{,ii}$: $\dfrac{\partial^2 \mathbf{H}}{\partial x_i^2}$

I : Area moment of inertia about y axis

$\mathbf{I}$: Identity matrix

$\mathbf{K}$: Global stiffness matrix

$\mathbf{K}_{NL}$: Nonlinear stiffness matrix

$\mathbf{K}_1^i,\ \mathbf{K}_2^i, \mathbf{K}_3^i$
$\mathbf{K}_4^i, \mathbf{K}_5^i, \mathbf{K}_6^i$: Elemental stiffness matrices

$\tilde{\mathbf{K}}$: Modal stiffness matrix

L : Mean Length

l_i : Length of ith element

$\mathbf{M}$: Global mass matrix

$\mathbf{M}_1^i, \mathbf{M}_2^i$: Elemental mass matrices

$\tilde{\mathbf{M}}$: Modal mass matrix

R : Radius

$\mathbf{R}$: Global internal and inertia force vector

$\mathbf{R}_i$: Elemental internal and inertia force vector

$\tilde{\mathbf{R}}$: Modal internal and inertia force vector

t : Time

T : Time period

$\tilde{T}$: Kinetic energy

u, w : Longitudinal and transverse displacements

u_0, w_0 : Longitudinal and transverse mid surface displacements

$\dot{u}$, $\dot{w}$: Longitudinal and transverse velocity

V : Potential energy

$w,_x$, $w,_{xx}$: $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}$

$w,_i$, $w,_{ii}$: $\dfrac{\partial w}{\partial x_i}, \dfrac{\partial^2 w}{\partial x_i^2}$

$w,_j$: $\dfrac{\partial w}{\partial x_j}$

$w,_s$: $\dfrac{\partial w}{\partial s}$

$\ddot{w}$: Transverse acceleration

$\dot{w},_x$: $\dfrac{\partial \dot{w}}{\partial x}$

w^i : Transverse displacement with in *i*th element

$\delta\tilde{T}$: First variation of kinetic energy

δV : First variation of potential energy

δW : Virtual work

x : Longitudinal coordinate

z : Transverse coordinate

Greek Notations

α, β : Rayleigh proportional damping parameter

δ : Variational symbol

θ : Rotation of normal in sz plane of curved beam

$\tilde{\theta}$: Angular position from excitation point in circular cylindrical shell

Δ : Increment in a quantity

$\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, \ddot{\boldsymbol{\eta}}$: Modal displacement, velocity, acceleration vectors

$\tilde{\boldsymbol{\eta}}$: State vector

$\tilde{\boldsymbol{\eta}}_0$: initial guess of state vector

υ : Poisson's ratio

ω_F : Forcing frequency

ω_i : Natural frequency of *i*th mode

ψ_i : Phase angle of *i*th harmonics

ρ : Density

$\mathbf{\Phi}$: Vector of eigen functions of linear free vibration modes of cantilever beam

$\mathbf{\Phi}_{,x}, \mathbf{\Phi}_{,xx}$: $\dfrac{\partial \mathbf{\Phi}}{\partial x}, \dfrac{\partial^2 \mathbf{\Phi}}{\partial x^2}$

ξ_1, ξ_2 : Damping factor in first anti-symmetric and symmetric modes of curved beams

$\lambda, \tilde{\delta}$: Constants of Newmark's time integration method