

EXPLORATIONS INTO THE ORLICZ FIGÀ-TALAMANCA HERZ ALGEBRA AND ITS DUAL SPACE

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EXPLORATIONS INTO THE ORLICZ FIGÀ-TALAMANCA HERZ ALGEBRA AND ITS DUAL SPACE

by

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Dedicated to
My Family

Certificate

This is to certify that the thesis entitled **Explorations into the Orlicz Figà-Talamanca Herz Algebra and its Dual Space**, submitted by **Mr. Arvish Dabra** to the **Indian Institute of Technology Delhi** for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis meets the standards and fulfills the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted, in part or in full, to any other university or institute for the award of any degree or diploma.

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“A PhD, like a Test match, is not about quick runs or flamboyant strokes; rather, it is a testament to patience, resilience, and the discipline of returning each day, even when the pitch is difficult.”

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Abstract

In this thesis, we investigate a range of structural and functorial properties of the *Orlicz Figà-Talamanca Herz algebra* $A_\Phi(G)$ and its dual space $PM_\Psi(G)$, associated with a *complementary pair of Young functions* (Φ, Ψ) satisfying the Δ_2 -condition.

We characterize the finite-dimensionality and reflexivity of the Banach algebra $A_\Phi(G)$ in terms of the underlying *locally compact group* G . We further examine the relationship between the Arens regularity of $A_\Phi(G)$ and the topological structure of G . In addition, we provide a criterion for the Arens regularity of $A_\Phi(G)$ formulated in terms of the spaces $PM_\Psi(G)$ and $PF_\Psi(G)$, where $PF_\Psi(G)$ denotes the algebra of Ψ -pseudofunctions. The Arens regularity of $PF_\Psi(G)$ is also studied in the setting of compact groups.

We then characterize the maximal regular left/right/two-sided ideals, as well as the minimal left ideals, of the algebras $\mathcal{A}^*$ (equipped with the Arens product) and $\mathcal{B}^{**}$, where $\mathcal{A}$ denotes a norm closed topologically introverted subspace of $PM_\Psi(G)$ and $\mathcal{B}$ denotes either $B_\Phi(G)$, the multiplier algebra of $A_\Phi(G)$, or $W_\Phi(G)$, the dual space of $PF_\Psi(G)$.

We investigate the relationship between certain topologically invariant subspaces or subalgebras of the *algebra of Ψ -pseudomeasures* $PM_\Psi(G)$ and the underlying group G . In this context, we obtain sufficient conditions for the amenability of G in terms of norm closed topologically invariant subspaces of $PM_\Psi(G)$. Moreover, under the assumption that Φ satisfies the MA condition, we establish a bijective correspondence between certain topologically invariant subalgebras of $PM_\Psi(G)$ and the closed subgroups of an amenable group G . An analogous result is also obtained for the predual $A_\Phi(G)$ in terms of compact subgroups of a locally compact group G .

We also study the topological center problem for both the algebra $\mathcal{A} = A_\Phi(G)$ and the associated space $\mathcal{A}^*\mathcal{A} = UCB_\Psi(\widehat{G})$. For an amenable group G , we show that the problem of determining the strong Arens irregularity of $A_\Phi(G)$ reduces to the case of open σ -compact subgroups. A similar reduction result is established for the topological center of the Banach algebra $UCB_\Psi(\widehat{G})^*$, equipped with the Arens product.

Finally, we investigate the isomorphism problem for the Ψ -convolution algebras $PM_\Psi(G)$ and $PF_\Psi(G)$, determining the extent to which the topological properties of the group G are reflected in these associated algebras. We conclude by proving an Orlicz analogue of Parrott's isomorphism theorem, thereby extending aspects of harmonic analysis to the Orlicz-space setting.

सार

इस शोध-प्रबंध में हम ऑर्लिकज़ फ़िगा-तालामांका हर्ज़ एल्जेब्रा $A_\Phi(G)$ तथा उससे संबद्ध इसके द्वैत स्थान $PM_\Psi(G)$ के विभिन्न संरचनात्मक एवं फंक्टोरियल गुणों का अध्ययन करते हैं। ये स्थान यंग फलनों की एक पूरक युग्म (Φ, Ψ) से संबंधित हैं, जो Δ_2 -शर्त को संतुष्ट करता है।

हम बानाख एल्जेब्रा $A_\Phi(G)$ की सीमित-आयामीता तथा रिफ़्लेक्सिविटी का वर्णन आधारभूत स्थानीय सघन समूह G के संदर्भ में करते हैं। इसके अतिरिक्त, हम $A_\Phi(G)$ की आरेन्स नियमितता और समूह G की टोपोलॉजिकल संरचना के बीच संबंध का विश्लेषण करते हैं। साथ ही, हम $PM_\Psi(G)$ तथा $PF_\Psi(G)$ के माध्यम से $A_\Phi(G)$ की आरेन्स नियमितता के लिए एक उपयुक्त मानदंड प्रस्तुत करते हैं, जहाँ $PF_\Psi(G)$, Ψ -छद्मफलनों की एल्जेब्रा को निरूपित करता है। इसके अतिरिक्त, हम संहत समूहों के ढाँचे में $PF_\Psi(G)$ की आरेन्स नियमितता का भी अध्ययन करते हैं।

आगे हम एल्जेब्राओं $\mathcal{A}^*$ (आरेन्स गुणन के अंतर्गत) तथा B^{**} के अधिकतम नियमित बाएँ, दाएँ एवं द्विपक्षीय आइडियल्स तथा न्यूनतम बाएँ आइडियल्स का चरित्रण करते हैं। यहाँ $\mathcal{A}$, $PM_\Psi(G)$ का एक नॉर्म-बंद टोपोलॉजिकल रूप से अंतर्मुख उपस्थान है, तथा B या तो $A_\Phi(G)$ की गुणक एल्जेब्रा $B_\Phi(G)$ है अथवा $PF_\Psi(G)$ का द्वैत स्थान $W_\Phi(G)$ है।

हम Ψ -छद्ममापों की एल्जेब्रा $PM_\Psi(G)$ के टोपोलॉजिकल रूप से अपरिवर्तनीय उपस्थानों तथा उपएल्जेब्राओं और आधारभूत समूह G के बीच संबंध का अध्ययन करते हैं। इसके अंतर्गत हम इन नॉर्म-बंद अपरिवर्तनीय उपस्थानों के माध्यम से समूह G की एमिनेबिलिटी के लिए पर्याप्त शर्तें प्राप्त करते हैं। इसके अतिरिक्त, यह मानते हुए कि Φ MA-शर्त को संतुष्ट करता है, हम $PM_\Psi(G)$ की कुछ टोपोलॉजिकल रूप से अपरिवर्तनीय उपएल्जेब्राओं और एक एमिनेबल समूह G के बंद उपसमूहों के बीच एक-एक अनुरूपता स्थापित करते हैं। साथ ही, स्थानीय सघन समूह G के संहत उपसमूहों के संदर्भ में उसके प्रीड्यूल $A_\Phi(G)$ के लिए भी एक समरूप परिणाम प्राप्त करते हैं।

हम एल्जेब्रा $\mathcal{A} = A_\Phi(G)$ तथा उससे संबद्ध स्थान $\mathcal{A}^*\mathcal{A} = UCB_\Psi(\hat{G})$ के लिए टोपोलॉजिकल केंद्र की समस्या का भी अध्ययन करते हैं। एक एमिनेबल समूह G के लिए हम यह सिद्ध करते हैं कि $A_\Phi(G)$ की रबल आरेन्स अनियमितता को निर्धारित करने की समस्या को खुले σ -संहत उपसमूहों के मामले तक सीमित किया जा सकता है। इसी प्रकार, आरेन्स गुणन के अंतर्गत परिभाषित बानाख एल्जेब्रा $UCB_\Psi(\hat{G})^*$ के टोपोलॉजिकल केंद्र के लिए भी समान अपचयन परिणाम सिद्ध किया गया है।

अंततः, हम Ψ -कन्वोल्यूशन एल्जेब्राओं $PM_\Psi(G)$ तथा $PF_\Psi(G)$ के लिए समरूपता समस्या का अध्ययन करते हैं तथा यह विश्लेषण करते हैं कि समूह G के टोपोलॉजिकल गुण किस सीमा तक इन संबद्ध एल्जेब्राओं में परिलक्षित होते हैं। अंत में, हम पैरॉट के समरूपता प्रमेय का एक ऑर्लिकज़ अनुरूप सिद्ध करते हैं, जिसके माध्यम से हार्मोनिक विश्लेषण को ऑर्लिकज़-स्थान के परिप्रेक्ष्य तक विस्तारित किया जाता है।

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List of Symbols

Symbol	Meaning
$\forall$	for all
$=$	equal to
$\neq$	not equal to
$\in$	belongs to
$\notin$	does not belong
$\subseteq$	subset or equal to
$\cup, \cap$	union, intersection
$\emptyset$	empty set
$\#X$	cardinality of the set X
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
Φ	Young function
G	locally compact Hausdorff group
$\mathcal{C}(G)$	the space of complex-valued continuous functions on G
$\mathcal{C}_c(G)$	the space of compactly supported continuous functions on G
$\mathcal{C}_0(G)$	the space of continuous functions on G vanishing at infinity
δ_a	the Dirac-delta measure at a in G
$\text{supp}(f)$	the support of a complex-valued function f on G

Abbreviation	Meaning
w.r.t.	with respect to
BAI	bounded approximate identity
RNP	Radon–Nikodým property
WSC	weakly sequentially complete

1

Introduction

1.1 Banach Algebras: A Brief History

The theory of *Banach algebras* has a rich history, with extensive research carried out over the past century. Among the most fundamental and thoroughly studied Banach algebras is $\mathcal{B}(E)$, the algebra consisting of all bounded linear operators on a Banach space E , equipped with composition as its multiplication.

Within the framework of *abstract harmonic analysis*, attention is devoted to those Banach algebras that arise from *locally compact groups*. Two of the most prominent examples in this context are the group algebra $L^1(G)$ and the measure algebra $M(G)$, associated with a locally compact group G . Moreover, several important operator algebras can be constructed from the group algebra $L^1(G)$, most notably the reduced group C^* -algebra $C_r^*(G)$ and the group von Neumann algebra $VN(G)$. These operator algebras form a cornerstone of *noncommutative harmonic analysis*.

A classical and celebrated result for the group algebra $L^1(G)$ was established by Wendel [79] (1951), who proved that $L^1(G)$ and $L^1(H)$ are isometrically isomorphic as Banach algebras if and only if the underlying groups G and H are topologically isomorphic. Subsequently, in 1964, Johnson [47] extended this line of research by obtaining an analogous isomorphism result for the measure algebra $M(G)$. In 1968, motivated by Wendel's celebrated result, Parrott [69, Theorem 2] proved that if there exists a surjective linear isometry

$$T : L^p(G) \rightarrow L^p(H), \quad p \in (1, \infty) \setminus \{2\},$$

such that $T(f * g) = T(f) * T(g)$ whenever f, g , and $f * g \in L^p(G)$, and similarly for T^{-1} , then G and H are isomorphic as topological groups.

A few years later, in his seminal 1972 monograph, Johnson [48] established one

of the most influential results in the theory of Banach algebras by proving that a locally compact group G is *amenable* if and only if the associated group algebra $L^1(G)$ is *amenable as a Banach algebra*. This result stands as a landmark in abstract harmonic analysis, forging a profound conceptual bridge between harmonic analysis, functional analysis, and topological group theory; a connection that has influenced decades of subsequent research.

It is well known that the group algebra $L^1(G)$ and the measure algebra $M(G)$ are *commutative* if and only if the underlying group G is *abelian*. Consequently, when G is *non-abelian*, the analysis of the algebras $L^1(G)$ and $M(G)$ becomes inherently constrained by their *noncommutative* structure. To address this limitation, Eymard [23] (1964) introduced the *Fourier algebra* $A(G)$ and the *Fourier–Stieltjes algebra* $B(G)$, which together provide a natural *commutative* Banach algebra framework associated with *any* locally compact group G .

The Fourier algebra $A(G)$ is defined by

$$A(G) := \{u = f * \check{g} : f, g \in L^2(G)\},$$

where $\check{g}(x) = g(x^{-1})$ for $x \in G$. For any $u \in A(G) \subseteq \mathcal{C}_0(G)$, the norm is given by

$$\|u\| := \inf \{ \|f\|_2 \|g\|_2 : u = f * \check{g}, f, g \in L^2(G) \}.$$

Equipped with this norm, together with pointwise addition and multiplication, $A(G)$ becomes a *commutative* Banach algebra. Moreover, $A(G)$ is semi-simple and regular as an algebra of continuous functions on G .

Eymard showed that the dual space of $A(G)$ can be identified isometrically with the group von Neumann algebra $VN(G)$. Moreover, when G is *abelian*, the Fourier algebra $A(G)$ coincides, up to an isometric isomorphism of Banach algebras, with the classical group algebra $L^1(\widehat{G})$, where $\widehat{G}$ denotes the Pontryagin dual of G . In this sense, $A(G)$ serves as a natural generalization of $L^1(\widehat{G})$ to the *non-abelian* setting.

In 1968, Leptin [60] provided a fundamental characterization of the amenability of a locally compact group G in terms of its Fourier algebra $A(G)$. Specifically, he proved that a group G is amenable if and only if the Banach algebra $A(G)$ admits a bounded approximate identity. This result established a profound connection between a purely algebraic property of $A(G)$ and the intrinsic harmonic-analytic structure of the underlying group G .

Despite this success, for several decades many attempts to extend Johnson’s celebrated theorem, characterizing group amenability via the amenability of $L^1(G)$, to the framework of Fourier algebra $A(G)$ were unsuccessful. A major breakthrough came in 1995, when Ruan [74] observed that, since $A(G)$ is the predual of the group von Neumann algebra $VN(G)$, it naturally inherits an *operator space structure*.

Consequently, $A(G)$ can be regarded not only as a Banach algebra but also as an *operator space*.

Ruan demonstrated that the pointwise multiplication on $A(G)$ is not merely continuous but is, in fact, *completely contractive* when endowed with this operator space structure. Building on this insight, he introduced the notion of *operator amenability* for completely contractive Banach algebras and established that $A(G)$ is operator amenable if and only if G is amenable. This result extended Johnson's celebrated theorem from the setting of $L^1(G)$ to the Fourier algebra $A(G)$, thereby unifying Banach algebra amenability and operator space theory.

The Fourier algebra $A(G)$ and its dual $VN(G)$ have been the subject of extensive investigation by numerous researchers, including Cecchini and Zappa [9], Civin and Yood [10], Forrest et al. [29], Lau [54], Lau and Losert [56], Lau and Ülger [57, 58], Lau and Wong [59], and Losert [62]. For further details and comprehensive discussions, the reader is referred to these works and the references cited therein.

Although the L^2 –space offers a rich Hilbert space structure that allows the use of geometric and spectral methods, many natural problems involve functions that fail to be square-integrable. To accommodate such cases, the L^p –framework generalizes the notion of integrability by replacing the quadratic norm with the more general p –norm, thereby yielding a family of Banach spaces $L^p(G)$ that capture a wider variety of functional behaviors depending on the value of p .

In 1965, Figà–Talamanca [24] introduced the p –analogue ($1 < p < \infty$) of the Fourier algebra $A(G)$ for a locally compact *abelian* group G , denoted by $A_p(G)$. Later, in 1971, Herz [41] extended this construction to *arbitrary* locally compact groups by employing tensor product techniques. The resulting algebra is denoted by $A_p(G)$ and is known as the *Figà–Talamanca Herz algebra*. This algebra generalizes Eymard's Fourier algebra $A(G) = A_2(G)$, thereby extending harmonic analysis from the Hilbert space setting to the more flexible Banach space framework.

The Figà–Talamanca Herz algebra $A_p(G)$ is defined as the set of all continuous functions $u \in \mathcal{C}_0(G)$ that can be represented in the form

$$u = \sum_{n \in \mathbb{N}} f_n * \check{g}_n,$$

where $f_n \in L^p(G)$ and $g_n \in L^{p'}(G)$, with $1 < p < \infty$ and p' denoting the conjugate exponent of p (i.e., $1/p + 1/p' = 1$), such that

$$\sum_{n \in \mathbb{N}} \|f_n\|_p \|g_n\|_{p'} < \infty.$$

For any $u \in A_p(G)$, the norm is defined by

$$\|u\|_{A_p} := \inf \left\{ \sum_{n \in \mathbb{N}} \|f_n\|_p \|g_n\|_{p'} : u = \sum_{n \in \mathbb{N}} f_n * \check{g}_n \right\}.$$

Equipped with this norm, together with pointwise addition and multiplication, $A_p(G)$ becomes a *commutative* Banach algebra. Moreover, $A_p(G)$ is semi-simple and regular as an algebra of continuous functions on G .

Let λ_p denote the *left regular representation* of the measure algebra $M(G)$ on $L^p(G)$, defined by

$$(\lambda_p(\mu))(f) := \mu * f, \quad \text{for all } \mu \in M(G), f \in L^p(G),$$

where convolution is taken with respect to the left Haar measure on the group G . The *algebra of p -pseudomeasures*, denoted by $PM_p(G)$, is defined as the w^* -closure of $\lambda_p(M(G))$ in the space of bounded linear operators $\mathcal{B}(L^p(G))$. Herz proved that the dual space of $A_p(G)$ is isometrically isomorphic to $PM_{p'}(G)$. In the special case $p = 2$, the algebra $PM_2(G)$ coincides with the group von Neumann algebra $VN(G)$.

In 1973, Herz [42] extended Leptin's characterization of the amenability of a locally compact group G , originally formulated in terms of the Fourier algebra $A(G)$, to the broader framework of the Figà-Talamanca Herz algebra $A_p(G)$. Specifically, he proved that a group G is amenable if and only if the Banach algebra $A_p(G)$ admits a bounded approximate identity.

In 2010, Daws [13] introduced the concept of a *p -operator space* to extend the operator space theory of the Fourier algebra to the setting of the Figà-Talamanca Herz algebra $A_p(G)$. Within this framework, he proved that $A_p(G)$ is *p -operator space amenable* if and only if the underlying group G is amenable. Although the *p -operator space* structure differs fundamentally from the classical operator space structure introduced by Ruan (which applies specifically to the Hilbert space case $p = 2$), it plays a crucial role in extending many results from the operator space theory of $A(G)$ to the more general context of $A_p(G)$.

The *algebra of p -convolution operators*, denoted by $CV_p(G)$ and the algebra of *p -pseudomeasures*, denoted by $PM_p(G)$, have been the focus of significant recent research by Daws and Spronk [14] as well as by Gardella and Thiel [30–32, 34]. These convolution algebras play a central role in the study of L^p -representations of locally compact groups, thereby generalizing the relationships among the group von Neumann algebra $VN(G)$, the group C^* -algebra $C^*(G)$ and the Fourier algebra $A(G)$ from the Hilbert space setting ($p = 2$) to the broader L^p -framework.

In a recent breakthrough, Gardella and Thiel [34] (2022) extended Wendel's classical isomorphism theorem to these L^p -convolution algebras. They proved that if $CV_p(G)$ and $CV_p(H)$ are isometrically isomorphic for $p \in [1, \infty) \setminus \{2\}$, then the

underlying groups G and H are topologically isomorphic. Moreover, they established an analogous result for the algebra $PM_p(G)$, showing that this operator–algebraic structure also completely determines the topology of the underlying group G .

The algebra $A_p(G)$ and its dual $PM_{p'}(G)$ have consequently attracted extensive attention from numerous researchers over the past several decades, including Daws [13], Derighetti [17–19], Forrest [27, 28], Granirer [36–39], Lau and Ülger [57], and Miao [64–67], among others. Their contributions explore various geometric and structural properties of these algebras, deepening the understanding of harmonic analysis on the L^p –spaces and the interplay between $A_p(G)$ and $PM_{p'}(G)$. For a comprehensive study, we refer the reader to the aforementioned works and the references cited therein.

It is well known that *Orlicz spaces* provide a broad and natural generalization of classical Lebesgue spaces. For a locally compact group G and a Young function Φ , the corresponding Orlicz space is denoted by $L^\Phi(G)$. In particular, this space coincides with $L^p(G)$ whenever $\Phi(x) = |x|^p/p$ for $1 \leq p < \infty$. Orlicz spaces have gained considerable attention due to their ability to exhibit structural and analytical properties that are absent in the Lebesgue spaces. For instance, when G is *compact*, one has

$$\bigcup_{\Phi \text{ is a } N\text{-function}} L^\Phi(G) = L^1(G), \quad \text{whereas} \quad \bigcup_{p>1} L^p(G) \subsetneq L^1(G).$$

Further, the Hardy–Littlewood maximal operator is bounded in certain Orlicz spaces $L^\Phi(\mathbb{T})$ for suitable Young functions Φ , whereas it fails to be bounded in the classical space $L^1(\mathbb{T})$ [49]. Moreover, Orlicz spaces include several important non–Lebesgue function classes, such as the exponential space, which plays a crucial role in the theory of Sobolev space embeddings, and the $L \log^+ L$ space, which provides a natural setting for studying the Hardy–Littlewood maximal operator. Consequently, these spaces have found wide–ranging applications in various areas of mathematical analysis, including nonlinear partial differential equations, stochastic analysis, and Fourier analysis [70–72].

Lal and Kumar [50] (2019), and independently Aghababa and Akbarbaglu [3] (2020), introduced and studied the Orlicz (L^Φ)–analogue of the Figà–Talamanca Herz algebra $A_p(G)$. Their construction arises by replacing the Lebesgue space $L^p(G)$ with the more general Orlicz space $L^\Phi(G)$, where Φ is a Young function satisfying the Δ_2 –condition ($\Phi \in \Delta_2$). The resulting algebra, denoted by $A_\Phi(G)$, is referred to as the *Orlicz Figà–Talamanca Herz algebra*. This algebra provides a natural extension of the classical Figà–Talamanca Herz algebra $A_p(G)$. Such a generalization enables the study of functions and operators under a wider range of growth conditions than those allowed by the power functions defining the L^p –spaces.

Formally, the Orlicz Figà–Talamanca Herz algebra $A_\Phi(G)$ is defined as the set of all continuous functions $u \in \mathcal{C}_0(G)$ that can be expressed in the form

$$u = \sum_{n \in \mathbb{N}} f_n * \check{g}_n,$$

where $f_n \in (L^\Phi(G), N_\Phi(\cdot))$ and $g_n \in (L^\Psi(G), \|\cdot\|_\Psi)$, with Ψ as the complementary Young function associated with Φ , such that

$$\sum_{n \in \mathbb{N}} N_\Phi(f_n) \|g_n\|_\Psi < \infty.$$

For any $u \in A_\Phi(G)$, the norm is given by

$$\|u\|_{A_\Phi} := \inf \left\{ \sum_{n \in \mathbb{N}} N_\Phi(f_n) \|g_n\|_\Psi : u = \sum_{n \in \mathbb{N}} f_n * \check{g}_n \right\}.$$

Equipped with this norm, together with pointwise addition and multiplication, $A_\Phi(G)$ becomes a *commutative* Banach algebra. Moreover, $A_\Phi(G)$ is semi-simple and regular as an algebra of continuous functions on G .

Let λ_Ψ denote the *left regular representation* of the measure algebra $M(G)$ on the Orlicz space $L^\Psi(G)$, defined by

$$(\lambda_\Psi(\mu))(f) := \mu * f, \quad \text{for all } \mu \in M(G), f \in L^\Psi(G),$$

where convolution is taken with respect to the left Haar measure on the group G . The *algebra of Ψ -pseudomeasures*, denoted by $PM_\Psi(G)$, is defined as the w^* -closure of $\lambda_\Psi(M(G))$ in the space of bounded linear operators $\mathcal{B}(L^\Psi(G))$. Lal and Kumar [50, Theorem 3.5] proved that the dual space of $A_\Phi(G)$ is isometrically isomorphic to $PM_\Psi(G)$.

In 2022, Lal and Kumar [52, Theorem 3.1] established that a group G is amenable if and only if the Banach algebra $A_\Phi(G)$ admits a bounded approximate identity, provided that the Young function Φ satisfies the MA condition. This result extends Herz's theorem for $A_p(G)$ to the more general setting of $A_\Phi(G)$. However, the bound of the approximate identity obtained by Lal and Kumar is 2, in contrast to the optimal bound 1 in the classical case of $A_p(G)$.

Aghababa and Akbarbaglu [3] introduced *the space of Φ -convolution operators*, denoted by $CV_\Phi(G)$, as the set of all bounded linear operators $T : L^\Phi(G) \rightarrow L^\Phi(G)$ satisfying

$$T(f * g) = T(f) * g, \quad \text{for all } f, g \in \mathcal{C}_c(G).$$

It follows that $CV_\Phi(G)$ is a closed subalgebra of $\mathcal{B}(L^\Phi(G))$ and consequently, is a unital Banach algebra under the usual operator composition. Moreover, it is well known that $PM_\Phi(G) \subseteq CV_\Phi(G)$ for any Young function Φ . In 2023, Tabatabaie and Latifpour [75] investigated the Orlicz (L^Φ) -version of the isomorphism problem.

They established that, for a certain class of Young functions Φ , if $CV_\Phi(G)$ and $CV_\Phi(H)$ are isometrically isomorphic, then the underlying groups G and H are topologically isomorphic.

For a comprehensive study of the Banach algebra $A_\Phi(G)$ and its dual $PM_\Psi(G)$, corresponding to a complementary pair of Young functions $(\Phi, \Psi) \in \Delta_2$, we refer the reader to the series of papers [3–5, 50–53] and the references therein.

1.2 Thesis Overview

In this thesis, we further investigate various structural and functorial properties of the Banach algebra $A_\Phi(G)$ and its dual space $PM_\Psi(G)$, which are associated with a complementary pair of Young functions (Φ, Ψ) satisfying the Δ_2 –condition.

Chapter 2 is devoted to the preliminary material. In Section 2.1, we provide the necessary background on the Orlicz space $L^\Phi(G)$ corresponding to a Young function Φ and a locally compact group G . Section 2.2 reviews the fundamental terminology, spaces, and algebras related to the Orlicz Figà–Talamanca Herz algebra $A_\Phi(G)$ and its dual space $PM_\Psi(G)$, associated with a complementary pair of Young functions (Φ, Ψ) satisfying the Δ_2 –condition. This section also summarizes the essential notations and preliminary results that will be employed throughout the thesis. Section 2.3 recalls the definition of the Arens product on a Banach algebra, together with the associated notations and terminology required in the subsequent chapters.

Chapter 3 focuses on the study of the Arens regularity of the algebra $A_\Phi(G)$ and several related structural properties. Section 3.1 begins by characterizing the finite–dimensionality and reflexivity of the Banach algebra $A_\Phi(G)$ in terms of the underlying group G . In particular, we establish that $A_\Phi(G)$ is finite–dimensional if and only if the group G itself is finite. Furthermore, when G is amenable and Φ satisfies the MA condition, we demonstrate that $A_\Phi(G)$ is reflexive precisely in this same situation; that is, if and only if G is finite.

In Section 3.2, attention is directed toward the connection between the Arens regularity of $A_\Phi(G)$ and the topological structure of the group G . We prove that the Arens regularity of $A_\Phi(G)$ necessarily implies that G is discrete. Furthermore, we provide an equivalent condition for the Arens regularity of $A_\Phi(G)$ formulated in terms of the spaces $PM_\Psi(G)$ and $PF_\Psi(G)$, where $PF_\Psi(G)$ is the algebra of Ψ –pseudofunctions. Section 3.3 continues this investigation under the assumption that the group G is discrete. The chapter concludes with Section 3.4, which focuses on the space $PF_\Psi(G)$. We show that the algebra $PF_\Psi(G)$ is Arens regular whenever G is compact, and that its dual $W_\Phi(G)$ exhibits the Radon–Nikodým property.

Chapter 4 is devoted to the study and characterization of the left/right/two-sided maximal regular ideals, as well as the minimal left ideals, of the Banach algebras $\mathcal{A}^*$ (equipped with the Arens product) and $\mathcal{B}^{**}$, where $\mathcal{B}$ denotes either $W_\Phi(G)$ or $B_\Phi(G)$, the multiplier algebra of $A_\Phi(G)$, and $\mathcal{A}$ denotes a norm closed topologically introverted subspace of $PM_\Psi(G)$. Specifically, the spaces under consideration include

$$C_{\delta,\Psi}(\widehat{G}), PF_\Psi(G), M(\widehat{G}), UCB_\Psi(\widehat{G}), AP_\Psi(\widehat{G}), WAP_\Psi(\widehat{G}), \text{ and } PM_\Psi(G).$$

In Section 4.1, we establish a characterization of the maximal regular ideals of $\mathcal{A}^*$ and $\mathcal{B}^{**}$. In particular, we show that if M is any maximal regular left/right/two-sided ideal of $\mathcal{A}^*$, then M is either weak*-dense in $\mathcal{A}^*$ or there exists a unique $x \in G$ such that

$$M = \{\Gamma \in \mathcal{A}^* : \langle \Gamma, \lambda_\Psi(x) \rangle = 0\}.$$

Section 4.2 concerns the structural properties of the minimal left ideals of $\mathcal{A}^*$. We prove that a left ideal M of $\mathcal{A}^*$ is minimal if and only if

$$M = k\Gamma,$$

where $\Gamma \in \mathcal{A}^*$ is either topologically x -invariant for some $x \in G$ or a non-zero right annihilator of $\mathcal{A}^*$. Furthermore, we demonstrate the existence of minimal ideals in the Banach algebras $A_\Phi(G)$ and $\mathcal{B}$.

Chapter 5 investigates the relationship between certain topologically invariant subspaces or subalgebras of the algebra of Ψ -pseudomeasures $PM_\Psi(G)$ and the underlying group G . In Section 5.1, we establish certain sufficient conditions for the amenability of G in terms of the norm closed topologically invariant subspaces of $PM_\Psi(G)$. This result provides a functional-analytic characterization of amenability through the internal structure of these subspaces.

In Section 5.2, assuming that the Young function Φ satisfies the MA condition, we establish a one-to-one correspondence between certain topologically invariant subalgebras of $PM_\Psi(G)$ and the closed subgroups of an amenable group G . In Section 5.3, we obtain a similar result for the predual $A_\Phi(G)$ and prove the existence of a bijective correspondence between certain topologically invariant subalgebras of $A_\Phi(G)$ and the compact subgroups of a locally compact group G .

Chapter 6 is devoted to the study of the topological center problem for the algebra $\mathcal{A} = A_\Phi(G)$ and the corresponding space $\mathcal{A}^*\mathcal{A} = UCB_\Psi(\widehat{G})$, extending several known results to a broader functional-analytic framework. Section 6.1 examines the existence of left and right identities in the double dual algebra $A_\Phi(G)^{**}$, equipped with the Arens product, and analyze their connection with the space $UCB_\Psi(\widehat{G})$. Furthermore, we provide a characterization of the amenability of the group G in terms of the existence of a right identity in $A_\Phi(G)^{**}$, under the assumption that the

Young function Φ satisfies the MA condition.

In Section 6.2, we establish sufficient conditions under which the restriction of an element of $A_\Phi(G)^{**}$ to $UCB_\Psi(\widehat{G})$ lies within $A_\Phi(G)$. This result plays a pivotal role in proving the main theorem of the chapter concerning the topological center of the double dual algebra $A_\Phi(G)^{**}$, denoted by $\Lambda(A_\Phi(G)^{**})$. Section 6.3 addresses the problem of strong Arens irregularity for $A_\Phi(G)$ in the setting of amenable groups. We demonstrate that, for such groups, this problem can be reduced to the case of open σ -compact subgroups. More precisely, we prove that

$$\Lambda(A_\Phi(G)^{**}) = A_\Phi(G) \quad \text{if and only if} \quad \Lambda(A_\Phi(H)^{**}) = A_\Phi(H)$$

for every open σ -compact subgroup H of an amenable group G , provided that the Young function Φ satisfies the MA condition. Furthermore, we obtain an analogous result for the topological center of the Banach algebra $UCB_\Psi(\widehat{G})^*$.

Chapter 7 investigates the isomorphism problem for the Ψ -convolution algebras $PM_\Psi(G)$ and $PF_\Psi(G)$. The primary objective of this chapter is to determine the extent to which the algebraic and topological structure of a locally compact group G is determined by the properties of these associated algebras. Sections 7.1 and 7.2 present isomorphism theorems for the algebras $PM_\Psi(G)$ and $PF_\Psi(G)$, respectively. In particular, for a certain class of Young functions Ψ , we show that if there exists an isometric isomorphism either between $PM_\Psi(G)$ and $PM_\Psi(H)$ or between $PF_\Psi(G)$ and $PF_\Psi(H)$, then the underlying groups G and H are topologically isomorphic. Finally, Section 7.3 extends this line of analysis by establishing an Orlicz analogue of Parrott's isomorphism theorem, thereby broadening the scope of harmonic analysis to the framework of Orlicz spaces.

Chapter 8 concludes this thesis by presenting several potential directions for future research that have emerged throughout the course of this study, highlighting areas that have particularly captured our interest and curiosity.

2

Preliminaries

This chapter presents the fundamental concepts, definitions, and notations that form the basis for the work discussed in the subsequent chapters. Its aim is to provide the necessary background for understanding the results presented later in this thesis.

2.1 Orlicz Spaces

In this section, we recall the fundamental terminology related to Orlicz spaces. An even convex function $\Phi : \mathbb{R} \rightarrow [0, \infty]$ is called a *Young function* if it satisfies

$$\Phi(0) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \Phi(x) = +\infty.$$

According to this definition, a Young function may take the value ∞ at certain points and therefore may be discontinuous.

If Φ is a Young function, then the associated convex function Ψ is defined as

$$\Psi(y) := \sup \{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

This function Ψ is also a Young function and is called the *complementary function* to Φ . The pair (Φ, Ψ) is referred to as a *complementary pair of Young functions* and satisfies the *Young inequality*

$$xy \leq \Phi(x) + \Psi(y), \quad \forall x, y \geq 0.$$

As a standard example, for $1 < p < \infty$, the function $\Phi(x) = |x|^p/p$ is a Young function whose complementary function is $\Psi(y) = |y|^{p'}/p'$, where $1/p + 1/p' = 1$.

Two Young functions Φ and $\tilde{\Phi}$ are said to be *equivalent* if there exists positive constants k_1, k_2 and $x_0 \geq 0$ such that

$$\Phi(k_1x) \leq \tilde{\Phi}(x) \leq \Phi(k_2x), \quad \forall x \geq x_0.$$

A *continuous* Young function Φ is said to be an *N-function* if it satisfies the following conditions:

$$\Phi(x) = 0 \text{ if and only if } x = 0, \quad \lim_{x \rightarrow 0} \frac{\Phi(x)}{x} = 0, \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} = +\infty,$$

while $\Phi(\mathbb{R}) \subseteq \mathbb{R}^+$.

Let G be a *locally compact group* with *identity element* e , equipped with a *left Haar measure* dx . A Young function Φ is said to satisfy the Δ_2 -condition (globally), denoted by $\Phi \in \Delta_2$ (or $\Phi \in \Delta_2$ globally), if there exists a constant $k > 0$ and $x_0 \geq 0$ such that

$$\Phi(2x) \leq k \Phi(x), \quad \forall x \geq x_0 \ (x_0 = 0)$$

[71, Definition 1, Pg. 22]. According to [71, Remark, Pg. 46], Φ is said to be Δ_2 -regular if this condition holds locally when the measure is finite and globally when the measure is infinite.

A pair of complementary Young functions (Φ, Ψ) is said to satisfy the Δ_2 -condition if both Φ and Ψ satisfy the condition individually. It is straightforward to verify that the family of Young functions $\Phi_{\alpha,p} : x \mapsto \alpha |x|^p$ for $p \geq 1$ and $\alpha > 0$, satisfies the Δ_2 -condition. Another example is

$$\Phi(x) = (1 + |x|) \log(1 + |x|) - |x|, \quad x \in \mathbb{R},$$

which also satisfies the Δ_2 -condition. However, its complementary function

$$\Psi(y) = e^{|y|} - |y| - 1, \quad y \in \mathbb{R},$$

does not satisfy the Δ_2 -condition.

A Young function Φ is said to satisfy the Δ' -condition, denoted by $\Phi \in \Delta'$, if there exist constants $k > 0$ and $x_0 \geq 0$ such that

$$\Phi(xy) \leq k \Phi(x) \Phi(y), \quad \forall x, y \geq x_0.$$

The family of Young functions $\Phi_{\alpha,p}(x) = \alpha |x|^p$, for $p \geq 1$ and $\alpha > 0$, satisfies both the Δ_2 -condition and the Δ' -condition.

A Young function Φ is said to satisfy the *Milnes-Akimonich* condition (hereafter referred to as the MA condition) if, for each $\epsilon > 0$, there exist constants $k_\epsilon > 1$ and $x_0(\epsilon) \geq 0$ such that

$$\Phi'((1 + \epsilon)x) \geq k_\epsilon \Phi'(x), \quad x \geq x_0(\epsilon),$$

where Φ' denotes either the left or right derivative of Φ . By [71, Theorem 1, Pg. 7], the Young function Φ admits both left and right derivatives at every point and these coincide except possibly on a countable set. An example of a Young function satisfying the MA condition is

$$\Psi(y) = e^{|y|} - |y| - 1, \quad y \in \mathbb{R}.$$

However, its complementary function

$$\Phi(x) = (1 + |x|) \log(1 + |x|) - |x|, \quad x \in \mathbb{R},$$

does not satisfy the MA condition.

It is easy to verify that the family of Young functions defined by

$$\Phi_\alpha(x) := |x|^\alpha (1 + |\log |x||), \quad \alpha > 1,$$

satisfies the MA condition with $k_\epsilon = (1 + \epsilon)^{\alpha-1}$ and $x_0(\epsilon) = 1$. Furthermore, the corresponding complementary pair of N -functions $(\Phi_\alpha, \Psi_\alpha)$ also satisfies the Δ_2 -condition [71, Corollary 4, Pg. 26].

Given a Young function Φ , the associated *Orlicz space* is denoted by $L^\Phi(G)$ and defined as

$$L^\Phi(G) := \left\{ f : G \rightarrow \mathbb{C} \mid f \text{ is measurable and } \int_G \Phi(\alpha |f|) < \infty \text{ for some } \alpha > 0 \right\}.$$

This space becomes a Banach space when equipped with the *Luxemburg norm* (also called the *Gauge norm*), defined by

$$N_\Phi(f) := \inf \left\{ k > 0 : \int_G \Phi \left(\frac{|f|}{k} \right) \leq 1 \right\}.$$

Let Ψ be the complementary function to Φ . The *Orlicz norm* $\|\cdot\|_\Phi$ on $L^\Phi(G)$ is defined by

$$\|f\|_\Phi := \sup \left\{ \int_G |fg| : g \in L^\Psi(G) \text{ and } \int_G \Psi(|g|) \leq 1 \right\}.$$

These two norms are equivalent; in fact, for every $f \in L^\Phi(G)$,

$$N_\Phi(f) \leq \|f\|_\Phi \leq 2 N_\Phi(f).$$

If the Young function Φ satisfies the Δ_2 -condition, then $\mathcal{C}_c(G)$ is dense in $L^\Phi(G)$. Furthermore, if the complementary function Ψ is continuous and satisfies $\Psi(x) = 0$ if and only $x = 0$, then *the dual of $(L^\Phi(G), N_\Phi(\cdot))$ is isometrically isomorphic to $(L^\Psi(G), \|\cdot\|_\Psi)$* [71, Corollary 9, Pg. 111]. In particular, if the complementary pair of Young functions (Φ, Ψ) satisfies the Δ_2 -condition, then $L^\Phi(G)$ is a *reflexive* Banach space [71, Theorem 10, Pg. 112].

Let G be a *compact group*. Then, for any $f \in L^1(G)$, the maps

$$g \mapsto f * g \text{ and } g \mapsto g * f$$

from $L^\Phi(G)$ into $L^\Phi(G)$ are compact. Moreover, $L^\Phi(G)$ admits an approximate identity that is bounded in the $\|\cdot\|_1$ -norm.

For further details on Orlicz spaces, we refer the reader to [70–72].

2.2 Orlicz Figà–Talamanca Herz Algebras

We begin this section by introducing the definition of the Orlicz Figà–Talamanca Herz algebra.

For any function $g : G \rightarrow \mathbb{C}$, define

$$\check{g}(x) := g(x^{-1}), \quad \forall x \in G.$$

The *Orlicz Figà–Talamanca Herz algebra* $A_\Phi(G)$ is the set of all continuous functions $u \in \mathcal{C}_0(G)$ that can be represented as

$$u = \sum_{n \in \mathbb{N}} f_n * \check{g}_n,$$

where $f_n \in (L^\Phi(G), N_\Phi(\cdot))$ and $g_n \in (L^\Psi(G), \|\cdot\|_\Psi)$, satisfying

$$\sum_{n \in \mathbb{N}} N_\Phi(f_n) \|g_n\|_\Psi < \infty.$$

For any $u \in A_\Phi(G)$, the norm is defined by

$$\|u\|_{A_\Phi} := \inf \left\{ \sum_{n \in \mathbb{N}} N_\Phi(f_n) \|g_n\|_\Psi : u = \sum_{n \in \mathbb{N}} f_n * \check{g}_n \right\}.$$

Equipped with this norm, together with pointwise addition and multiplication, $A_\Phi(G)$ becomes a *commutative Banach algebra*. Moreover, it is a regular, Tauberian, and semi-simple Banach algebra whose *Gelfand spectrum is homeomorphic to G* .

From [50, Lemma 4.1], it follows that for any *open* subgroup H of G , the Banach algebra $A_\Phi(H)$ is isometrically isomorphic to the subalgebra of $A_\Phi(G)$ consisting of functions that vanish outside H . We denote by i_H the *inclusion map* from $A_\Phi(H)$ into $A_\Phi(G)$, and by m_H the *restriction map* from $A_\Phi(G)$ onto $A_\Phi(H)$. It follows from [50, Lemma 4.5] that m_H is a *contractive homomorphism* of Banach algebras. Moreover, using the above identification, one can easily verify that $A_\Phi(H)$ is a *closed ideal* of $A_\Phi(G)$. In addition, the map m_H is a *bounded projection* which also acts as a *multiplier*, i.e.,

$$m_H(uv) = u m_H(v) = m_H(u) v, \quad \forall u, v \in A_\Phi(G),$$

and

$$m_H(u) = u, \quad \forall u \in A_\Phi(H).$$

Let $B_\Phi(G)$ denote the *multiplier algebra* of $A_\Phi(G)$, defined by

$$B_\Phi(G) := \{u \in \mathcal{C}(G) : uv \in A_\Phi(G) \text{ for all } v \in A_\Phi(G)\}.$$

Each element $u \in B_\Phi(G)$ induces a bounded linear operator

$$M_u : A_\Phi(G) \rightarrow A_\Phi(G), \quad M_u(v) := uv, \quad \forall v \in A_\Phi(G).$$

Equipped with the operator norm and pointwise addition and multiplication, $B_\Phi(G)$ forms a *commutative Banach algebra*. Moreover, it is clear that

$$A_\Phi(G) \subseteq B_\Phi(G).$$

Let $M(G)$ be the measure algebra on a locally compact group G . For $\mu \in M(G)$, define the *left convolution operator*

$$\lambda_\Psi(\mu) : L^\Psi(G) \rightarrow L^\Psi(G)$$

by

$$(\lambda_\Psi(\mu))(g) := \mu * g, \quad g \in L^\Psi(G),$$

where

$$(\mu * g)(x) := \int_G g(y^{-1}x) d\mu(y).$$

Let $\mathcal{B}(L^\Psi(G))$ denote the Banach space of all bounded linear operators on $L^\Psi(G)$, equipped with the operator norm. It is easy to verify that $\lambda_\Psi(\mu) \in \mathcal{B}(L^\Psi(G))$ for every $\mu \in M(G)$.

Define $PM_\Psi(G)$ as the closure of the set

$$\{\lambda_\Psi(\mu) : \mu \in M(G)\}$$

in

$$\mathcal{B}(L^\Psi(G)) \cong (L^\Phi(G) \widehat{\otimes} L^\Psi(G))^* \quad (2.1)$$

with respect to the *ultra-weak topology* (i.e., the w^* -topology), where $\widehat{\otimes}$ denotes the projective tensor product. The identification (2.1) follows from [76, Theorem 2.3]. Aghababa and Akbarbaglu proved that the *algebra of Ψ -pseudomeasures* $PM_\Psi(G)$ is isometrically isomorphic to the dual space of the Orlicz Figà–Talamanca Herz algebra $A_\Phi(G)$ [3, Corollary 4.11].

If μ_f denotes the measure associated with $f \in L^1(G)$, then the *norm closure* of the set

$$\{\lambda_\Psi(f) := \lambda_\Psi(\mu_f) : f \in L^1(G)\}$$

in $\mathcal{B}(L^\Psi(G))$ is denoted by $PF_\Psi(G)$ and is called the *algebra of Ψ -pseudofunctions*. It is easy to verify that $PM_\Psi(G)$ is the w^* -closure of $PF_\Psi(G)$ in $\mathcal{B}(L^\Psi(G))$. The dual of $PF_\Psi(G)$ is $W_\Phi(G)$, where $W_\Phi(G)$ is a *commutative Banach algebra* containing $A_\Phi(G)$. By [53, Theorem 3.1], an element $u \in L^\infty(G)$ belongs to $W_\Phi(G)$ with $\|u\|_{W_\Phi(G)} \leq l$ if and only if, for every compact subset $K \subseteq G$, the restriction $u|_K$ belongs to $A_\Phi(K)$ and satisfies $\|u\|_{A_\Phi(K)} \leq l$.

Moreover, for $\mu = \delta_a$ (the Dirac–delta measure at $a \in G$), the corresponding operator $\lambda_\Psi(\delta_a)$ is denoted by $\lambda_\Psi(a)$. Thus, the map λ_Ψ defines the *left regular representation* of G on $L^\Psi(G)$. The fact that λ_Ψ is a contraction follows from [3,

Proposition 3.2]. Furthermore, by [3, Proposition 3.6], λ_Ψ is a faithful representation of the measure algebra $M(G)$.

We now recall several subspaces of $PM_\Psi(G)$ that are frequently used throughout this thesis:

$$\begin{aligned} C_{\delta,\Psi}(\widehat{G}) &:= \overline{\text{span}\{\lambda_\Psi(x) : x \in G\}}^{PM_\Psi(G)}, \\ M(\widehat{G}) &:= \overline{M(G)}^{PM_\Psi(G)}, \\ UCB_\Psi(\widehat{G}) &:= \overline{A_\Phi(G) \cdot PM_\Psi(G)}^{PM_\Psi(G)}, \\ AP_\Psi(\widehat{G}) &:= \{T \in PM_\Psi(G) : A_\Phi(G) \rightarrow PM_\Psi(G), u \mapsto u \cdot T \text{ is compact}\}, \\ WAP_\Psi(\widehat{G}) &:= \{T \in PM_\Psi(G) : A_\Phi(G) \rightarrow PM_\Psi(G), u \mapsto u \cdot T \text{ is weakly compact}\}. \end{aligned}$$

These subspaces are Orlicz analogues of the corresponding subspaces of $PM_p(G)$ considered in [38]. Moreover, we emphasize that the notation $\widehat{G}$ used here does not denote the Pontryagin dual of G , which is defined only for locally compact *abelian* groups. Rather, this notation follows Granirer [36], where $UCB(\widehat{G})$ denotes the subspace of uniformly continuous functionals on the Fourier algebra $A(G)$ for an arbitrary locally compact group G . In the special case where G is a locally compact *abelian* group, $A(G)$ is isometrically isomorphic to $L^1(\widehat{G})$ and $UCB(\widehat{G})$ coincides with the space of bounded uniformly continuous complex-valued functions on the Pontryagin dual $\widehat{G}$ of G .

Aghababa and Akbarbaglu [3] introduced *the space of Φ -convolution operators*, denoted by $CV_\Phi(G)$, as the set of all bounded linear operators

$$T : L^\Phi(G) \rightarrow L^\Phi(G)$$

that satisfy

$$T(f * g) = T(f) * g,$$

for all $f, g \in \mathcal{C}_c(G)$. It follows that $CV_\Phi(G)$ is a closed subalgebra of $\mathcal{B}(L^\Phi(G))$, and hence a unital Banach algebra under composition.

By [3, Theorem 5.5], an operator $T \in \mathcal{B}(L^\Phi(G))$ belongs to $CV_\Phi(G)$ if and only if it commutes with *right translations*, denoted by ρ_a ($a \in G$), i.e.,

$$T(\rho_a(f)) = \rho_a(T(f)),$$

for every $a \in G$ and $f \in L^\Phi(G)$, where $(\rho_a(f))(x) := f(xa)$ for $x \in G$. Therefore, the definition of $CV_\Phi(G)$ given by Aghababa and Akbarbaglu is equivalent to that introduced by Tabatabaie and Latifpour [75]. It is well known that the following inclusions hold for any Young function Φ :

$$PF_\Phi(G) \subseteq PM_\Phi(G) \subseteq CV_\Phi(G).$$

A linear functional $\Gamma \in PM_\Psi(G)^*$ is called a *topologically invariant mean* if it

satisfies

$$\|\Gamma\| = 1 = \langle \Gamma, \lambda_\Psi(e) \rangle \quad \text{and} \quad \langle \Gamma, u \cdot T \rangle = u(e) \langle \Gamma, T \rangle,$$

for all $u \in A_\Phi(G)$ and $T \in PM_\Psi(G)$. The set of all topologically invariant means on $PM_\Psi(G)$ is denoted by $TIM_\Psi(\widehat{G})$.

For further details on the Banach algebra $A_\Phi(G)$ and the associated spaces, we refer the reader to the series of papers [3–5, 50–53].

2.3 Arens Product and Introverted Subspaces

Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{A}^{**}$ denote its double dual. It is well known that $\mathcal{A}^{**}$ can be equipped with two natural extensions of the multiplication on $\mathcal{A}$, both of which turn $\mathcal{A}^{**}$ into a Banach algebra. These extensions, introduced by Arens [6], are commonly referred to as the *first* and *second Arens products*.

These products are defined as follows:

1. The *first Arens product*, denoted by $\square$:

- (a) $\langle u \cdot T, v \rangle = \langle T, uv \rangle$, for all $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.
- (b) $\langle \Gamma \odot T, u \rangle = \langle \Gamma, u \cdot T \rangle$, for all $u \in \mathcal{A}$, $T \in \mathcal{A}^*$ and $\Gamma \in \mathcal{A}^{**}$.
- (c) $\langle \tilde{\Gamma} \square \Gamma, T \rangle = \langle \tilde{\Gamma}, \Gamma \odot T \rangle$, for all $T \in \mathcal{A}^*$ and $\tilde{\Gamma}, \Gamma \in \mathcal{A}^{**}$.

2. The *second Arens product*, denoted by $\diamond$:

- (a) $\langle T \cdot u, v \rangle = \langle T, vu \rangle$, for all $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.
- (b) $\langle \tilde{\Gamma} \odot T, u \rangle = \langle \tilde{\Gamma}, T \cdot u \rangle$, for all $u \in \mathcal{A}$, $T \in \mathcal{A}^*$ and $\tilde{\Gamma} \in \mathcal{A}^{**}$.
- (c) $\langle \tilde{\Gamma} \diamond \Gamma, T \rangle = \langle \Gamma, \tilde{\Gamma} \odot T \rangle$, for all $T \in \mathcal{A}^*$ and $\tilde{\Gamma}, \Gamma \in \mathcal{A}^{**}$.

In general, $\tilde{\Gamma} \square \Gamma$ need not equal $\tilde{\Gamma} \diamond \Gamma$ for all $\tilde{\Gamma}, \Gamma \in \mathcal{A}^{**}$. If these two products coincide for all such pairs, then $\mathcal{A}$ is said to be *Arens regular*.

If $\mathcal{A}$ is a *commutative* Banach algebra, then $u \cdot T = T \cdot u$ for all $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$. Furthermore, $\mathcal{A}$ is Arens regular if and only if $\mathcal{A}^{**}$ is a *commutative* Banach algebra with respect to either (and hence both) of the Arens products [21, Proposition 1].

Let X be a closed linear subspace of $\mathcal{A}^*$. The subspace X is said to be *left topologically invariant* if

$$u \cdot T \in X \quad \text{for all } u \in \mathcal{A} \text{ and } T \in X.$$

Given such a subspace X of $\mathcal{A}^*$, if

$$\Gamma \odot T \in X \quad \text{for all } T \in X \text{ and } \Gamma \in X^*,$$

then X is called a *left topologically introverted* subspace of $\mathcal{A}^*$.

Similarly, one can define *right topologically introverted* subspace of $\mathcal{A}^*$. However, in our setting, since $\mathcal{A} = A_\Phi(G)$ is a *commutative* Banach algebra, *both notions coincide*.

For the reader's convenience, we recall a number of definitions that are used extensively in this thesis.

- A Banach algebra is said to be *weakly sequentially complete* (WSC) if every weakly Cauchy sequence converges with respect to the weak topology.
- A proper left ideal I of an algebra $\mathcal{A}$ is *regular* (or *modular*) [12, Pg. 40] if there exists an element $u \in \mathcal{A}$ such that

$$v - vu \in I \quad \text{for all } v \in \mathcal{A}.$$

In particular, a proper ideal I is regular if and only if $\mathcal{A}/I$ is a unital algebra. Note that if $\mathcal{A}$ itself is unital, then every ideal of $\mathcal{A}$ is regular.

- A Banach space X is said to have the *Radon–Nikodým property* (RNP) [8] if each closed convex subset D of X is dentable, i.e., for any $\epsilon > 0$, there exists $x \in D$ such that

$$x \notin \overline{\text{co}}(D \setminus B_\epsilon(x)),$$

where $B_\epsilon(x) = \{y \in X : \|x - y\| < \epsilon\}$.

- A locally compact group G is said to be *amenable* if there exists a left-invariant mean on $L^\infty(G)$, i.e., a positive linear functional Λ on $L^\infty(G)$ of norm one satisfying

$$\langle \Lambda, L_a(f) \rangle = \langle \Lambda, f \rangle \quad \text{for all } f \in L^\infty(G) \text{ and } a \in G,$$

where $L_a(f) : G \rightarrow \mathbb{C}$ is defined by $(L_a(f))(x) := f(a^{-1}x)$ for $x \in G$ and denotes the *left translation* of f .

Classical examples of amenable groups include abelian groups, compact groups, and solvable groups. In contrast, the free group on two generators $\mathbb{F}_2$ is a well-known example of a non-amenable group.

Throughout this thesis, we assume that G is a locally compact group with identity element e , equipped with a left Haar measure dx . Furthermore, the complementary pair of N -functions (Φ, Ψ) satisfies the Δ_2 -condition.

3

Arens Regularity and Related Results

Arens regularity is an important tool in the study of groups through certain associated Banach algebras. Arens [6] introduced two natural products on the double dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$, known as the *first* and *second* Arens products. A Banach algebra $\mathcal{A}$ is said to be *Arens regular* if these two products coincide. Civin and Yood [10] proved that if G is an *infinite* locally compact *abelian* group, then the group algebra $L^1(G)$ is *not* Arens regular. Subsequently, Young [80] established that for *any* locally compact group G , the algebra $L^1(G)$ is *Arens regular if and only if G is finite*. This result was later generalized to the Fourier algebra $A(G)$ by Lau and Wong [59], under the assumption that the group G is *amenable*. Recently, in 2023, Forrest et al. [29] studied the Arens regularity of ideals in $A(G)$ and related algebras.

In 1991, Forrest [27] showed that if the algebra $A_p(G)$ is *Arens regular*, then the underlying group G must be *discrete*. He later established that for an *abelian* group G , the algebra $A_p(G)$ is *Arens regular if and only if G is finite* [28]. In 1993, Lau and Ülger [57] provided an equivalent characterization of the Arens regularity of $A_p(G)$ in terms of the spaces $PM_p(G)$ and $PF_p(G)$.

A classical problem in functional analysis is to characterize those Banach spaces that are reflexive. Assuming that the group G is *amenable*, Granirer [37] proved that the algebras $A_p(G)$ are *reflexive if and only if G is finite*. In Section 3.1, we establish an analogous result for the algebras $A_\Phi(G)$ and also characterize when these algebras are finite-dimensional. Specifically, we show that $A_\Phi(G)$ is finite-dimensional if and only if G is finite.

In Section 3.2, we investigate the structure of the second dual $A_\Phi(G)^{**}$ as a Banach algebra with respect to the two Arens products. In particular, we show that if $A_\Phi(G)$ is Arens regular, then the underlying group G must be discrete.

Furthermore, we provide an equivalent condition for the Arens regularity of $A_\Phi(G)$ in terms of the spaces $PM_\Psi(G)$ and $PF_\Psi(G)$. The proof follows an approach inspired by the methods of Lau and Ülger [57].

In Section 3.3, we study the Arens regularity of $A_\Phi(G)$ under the assumption that the group G is discrete. Finally, in Section 3.4, we show that the algebra $PF_\Psi(G)$ is Arens regular whenever G is compact and that its dual $W_\Phi(G)$ possesses the Radon–Nikodým property.

3.1 Finite–Dimensionality and Reflexivity

In this section, we establish the necessary and sufficient conditions for the finite–dimensionality and reflexivity of the algebra $A_\Phi(G)$ in terms of the underlying group G . To achieve this, we first characterize the reflexive $A_\Phi(G)$ –submodules of $PM_\Psi(G)$. The results are inspired from Granirer’s work [37, 39].

Recall from [51, Proposition 3.3, Pg. 5] that the measure algebra $M(G)$ embeds into $PM_\Psi(G)$ via the following identification:

$$\mu \mapsto \lambda_\Psi(\mu),$$

where

$$(\lambda_\Psi(\mu))(v) := \int_G v \, d\mu,$$

for $v \in A_\Phi(G)$, or equivalently,

$$(\lambda_\Psi(\mu))(g) := \mu * g,$$

for $g \in L^\Psi(G)$.

Remark 3.1.1. For any $T \in PF_\Psi(G)^* = W_\Phi(G)$, we can define a bounded linear functional φ_T on $L^1(G)$ by

$$\varphi_T(f) := T(\lambda_\Psi(f)), \quad f \in L^1(G).$$

Then

$$|\varphi_T(f)| = |T(\lambda_\Psi(f))| \leq \|T\|_{PF_\Psi(G)^*} \|\lambda_\Psi(f)\|_{PF_\Psi(G)} \leq \|T\|_{W_\Phi(G)} \|f\|_1,$$

i.e., $\varphi_T \in L^1(G)^*$. Therefore, there exists $h_T \in \ell^\infty(G)$ such that

$$\varphi_T(f) = \int_G f(x) h_T(x) \, dx.$$

From the above computations, we have

$$\left| \int_G f h_T \right| \leq \|h_T\|_{W_\Phi(G)} \|f\|_{PF_\Psi(G)}.$$

This explains how we identify each element of $W_\Phi(G)$ as a function.

We begin with the following significant lemma.