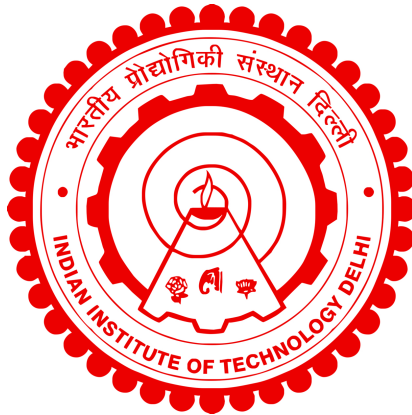


INCREMENTAL STABILITY BASED DESIGN AND ANALYSIS OF NONLINEAR SYSTEMS

BHABANI SHANKAR DEY



DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY DELHI

OCTOBER 2024

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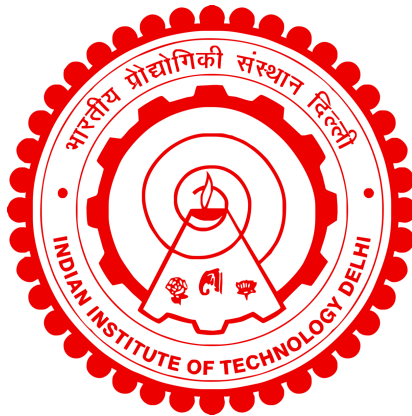
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Submitted

in partial fulfillment of the requirements of the degree of Doctor of Philosophy

to the



**INDIAN INSTITUTE OF TECHNOLOGY
DELHI**

OCTOBER 2024

Dedicated to family and friends...

Certificate

This is to certify that the thesis entitled “**Incremental Stability based Design and Analysis of Nonlinear Systems**”, submitted by **Bhabani Shankar Dey** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** in Electrical Engineering, is a record of the original, bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree.

The results contained in this thesis have not been submitted in part or in full to any other University or Institute for the award of any degree or diploma to the best of my knowledge.

Prof. Indra Narayan Kar

Department of Electrical Engineering,
Indian Institute of Technology Delhi.
(Supervisor)

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Abstract

In recent times, incremental stability has been explored to address various challenges such as tracking, regulation, observer design, coordination, and synchronization. Contraction analysis, a method leveraging the stability properties of linearized dynamics, is typically employed to infer incremental stability for nonlinear systems. In contrast to conventional Lyapunov stability analysis, contraction analysis prioritizes the convergence characteristics of trajectories with respect to each other instead of exclusively concentrating on a specific equilibrium point. This property is exploited in the observer, controller design, and synchronization problems in several applications. In this thesis, we have used contraction analysis for observer and controller design for several classes of systems based on easily tractable matrix measures conditions. The work also introduces output feedback control based on an extended high-gain observer (HGO), ensuring the contraction of a singularly perturbed (SP) system's contraction. Two main distinctions between this analysis and conventional Lyapunov techniques are that proving contraction does not require any interconnection conditions, and the convergence bounds explicitly rely on the singular perturbation parameter (SPP). Furthermore, it is noted that the requirement for the smallness of the SPP can be relaxed for certain types of nonlinear systems. The derived bounds also highlight a trade-off between the contraction rates and the condition numbers associated with the transformation metric. These bounds provide additional freedom for tuning closed-loop performance, aside from reducing the SPP. The standard contraction theory demands the dynamics to be sufficiently smooth. However, the smoothness assumption fails in many applications, such as switched and hybrid systems. There are very few existing literature that considers this issue, and it lacks comprehensive coverage of switched dynamical systems. This thesis introduces contraction analysis tailored for switched interconnected systems and derives sufficient conditions to ensure incremental stability. The methodology incorporates matrix measure-based analysis and extensive use of solution estimates of nonlinear dynamical systems. In the case of state-dependent switched systems, the key role of contraction theory lies in facilitating the regularization of the system by approximating it with a smooth dynamical system. This regularization process is essential for applying standard contraction results. The same idea of analysis of switched systems is extended to more practical interconnected switched nonlinear

systems in the presence of input. This has led to the introduction of the incremental ISS of the interconnected system. The small gain characterization of nonlinear systems connected in feedback plays a major role in interconnection. The analytical frameworks presented in the thesis cover that aspect as well. The results are derived for different classes of interconnection. The thesis also studies nonlinear switched systems based on time-dependent switching. The role of dwell time and average dwell time in time-dependent switched systems is discussed in detail. The incremental stability-based analysis and design are also extended to switching controllers, where switching is introduced in the system by virtue of controllers. At last, an interesting application in which harmonic suppression in switched mechanical systems is presented using contraction analysis.

सार

हाल के दिनों में, ट्रेकिंग, विनियमन, पर्यवेक्षक बनावट, समन्वय और समकालीकरण जैसी विभिन्न चुनौतियों का समाधान करने के लिए वार्षिक स्थिरता का पता लगाया गया है। संकुचन विश्लेषण, रैखिकृत गतिशीलता की स्थिरता गुणों का लाभ उठाने वाली एक विधि, आमतौर पर अरेखीय प्रणालियों के लिए वार्षिक स्थिरता का अनुमान लगाने के लिए नियोजित की जाती है। पारंपरिक लायपुनोव स्थिरता विश्लेषण के विपरीत, संकुचन विश्लेषण विशेष रूप से एक विशिष्ट संतुलन बिंदु पर ध्यान केंद्रित करने के बजाय एक दूसरे के संबंध में प्रक्षेपवक्र की अभिसरण विशेषताओं को प्राथमिकता देता है। इस गुण का उपयोग कई अनुप्रयोगों में पर्यवेक्षक, नियंत्रक डिज़ाइन और समकालीकरण समस्याओं में किया जाता है। इस थीसिस में, हमने आसानी से ट्रेक्टेबल मैट्रिक्स माप स्थितियों के आधार पर सिस्टम के कई वर्गों के लिए पर्यवेक्षक और नियंत्रक डिज़ाइन के लिए संकुचन विश्लेषण का उपयोग किया है। कार्य एक विस्तारित उच्च-लाभ पर्यवेक्षक के आधार पर आउटपुट फीडबैक नियंत्रण भी पेश करता है, जो एक विलक्षण रूप से परेशान प्रणाली के संकुचन को सुनिश्चित करता है। इस विश्लेषण और पारंपरिक ल्यपुनोव तकनीकों के बीच दो मुख्य अंतर यह हैं कि संकुचन को साबित करने के लिए किसी अंतर्संबंध की स्थिति की आवश्यकता नहीं होती है, और अभिसरण सीमाएँ स्पष्ट रूप से एकवचन गड़बड़ी पैरामीटर पर निर्भर करती हैं। इसके अलावा, यह ध्यान दिया गया है कि कुछ प्रकार के अरेखीय सिस्टम के लिए एसपीपी की छोटीता की आवश्यकता में छूट दी जा सकती है। व्युत्पन्न सीमाएँ संकुचन दरों और परिवर्तन मीट्रिक से जुड़ी स्थिति संख्याओं के बीच एक व्यापार-बंद को भी उजागर करती हैं। ये सीमाएँ एसपीपी को कम करने के अलावा, बंद-लूप प्रदर्शन को ट्यून करने के लिए अतिरिक्त स्वतंत्रता प्रदान करती हैं। मानक संकुचन सिद्धांत गतिशीलता को पर्याप्त रूप से सुचारू बनाने की मांग करता है। हालाँकि, सुचारूता धारणा कई अनुप्रयोगों में विफल हो जाती है, जैसे कि स्विच और हाइब्रिड सिस्टम। बहुत कम मौजूदा साहित्य है जो इस मुद्दे पर विचार करते हैं, और इसमें स्विच डायनेमिक सिस्टम के व्यापक कवरेज का अभाव है। यह थीसिस स्विच इंटरकनेक्टेड सिस्टम के लिए तैयार संकुचन विश्लेषण का परिचय देती है और वार्षिक स्थिरता सुनिश्चित करने के लिए पर्याप्त स्थितियाँ प्राप्त करती है। कार्यप्रणाली में मैट्रिक्स माप-आधारित विश्लेषण और अरेखीय गतिशील प्रणालियों के समाधान अनुमानों का व्यापक उपयोग शामिल है। राज्य-निर्भर स्विच सिस्टम के मामले में, संकुचन सिद्धांत की मुख्य भूमिका एक सुचारू गतिशील प्रणाली के साथ सिस्टम को नियमित करने की सुविधा प्रदान करने में निहित है। मानक संकुचन परिणामों को लागू करने के लिए यह नियमितीकरण प्रक्रिया आवश्यक है। स्विच सिस्टम के विश्लेषण का यही विचार इनपुट की उपस्थिति में अधिक व्यावहारिक इंटरकनेक्टेड स्विच नॉनलाइनियर सिस्टम तक विस्तारित है। इससे इंटरकनेक्टेड सिस्टम के वृद्धिशील आईएसएस की शुरुआत हुई है। फीडबैक में जुड़े अरेखीय प्रणाली का छोटा लाभ लक्षण वर्णन इंटरकनेक्शन में एक प्रमुख भूमिका निभाता है। थीसिस में प्रस्तुत विश्लेषणात्मक रूपरेखा उस पहलू को भी कवर करती है। परिणाम अंतरसंबंध के विभिन्न वर्गों के लिए निकाले गए हैं। थीसिस समय-निर्भर स्विचिंग के आधार पर अरेखीय स्विच सिस्टम का भी अध्ययन करती है। समय-निर्भर स्विच सिस्टम में रुकने के समय और औसत रुकने के समय की भूमिका पर विस्तार से चर्चा की गई है। वार्षिक स्थिरता-आधारित विश्लेषण और डिज़ाइन को स्विचिंग

नियंत्रकों तक भी बढ़ाया जाता है, जहां नियंत्रकों के आधार पर सिस्टम में स्विचिंग शुरू की जाती है। अंत में, एक दिलचस्प एप्लिकेशन जिसमें स्विच मैकेनिकल सिस्टम में हार्मोनिक दमन संकुचन विश्लेषण का उपयोग करके प्रस्तुत किया गया है।

Contents

Certificate	i
Acknowledgements	ii
Abstract	iii
Contents	vii
List of Figures	xi
List of Tables	xv
Abbreviations	xvii
Symbols	xix
1 Introduction	1
1.1 Objective of the Thesis	4
1.2 Brief Literature Review	5
1.3 Issues and Directions of the Thesis	6
1.3.1 Contraction theory based Observer and Controller Design for Artificial Pancreas	6
1.3.2 Output Feedback Control for Nonlinear SPS using Contraction Analysis	8
1.3.3 Incremental Stability of Switched Interconnected System	9
1.4 Preliminaries and Background	11
1.4.1 Incremental Stability	11
1.4.2 Incremental ISS	12
1.4.3 Convergent Systems	15
1.4.4 Incremental Stability and Contraction	18

1.4.5	A Revisit to Contraction Theory	19
1.4.5.1	Proof of Contraction	20
1.4.5.2	An Alternate Proof of the Contraction	22
1.4.5.3	Virtual System and Partial Contraction Analysis	24
1.4.6	Incremental Stability of Switched Systems	27
1.4.6.1	Contraction Analysis of Differential Equations with Discontinuous Right Hand Side	28
1.5	Scope and Organization of the Thesis	32
2	Observer and Controller Design using Contraction Theory	39
2.1	Introduction	39
2.2	Methods	43
2.2.1	Mathematical Model	43
2.2.1.1	Meal Disturbance Model	45
2.2.2	Preliminaries	46
2.3	Observer Design for Artificial Pancreas	47
2.3.1	Estimation Results:	50
2.4	Controller Design for Artificial Pancreas	52
2.5	Simulation Study	57
2.5.1	Scenario 1: A single day, three meal scenario with nominal parameters	57
2.5.2	Scenario 2: A single day, 3 meal scenario with intra-patient variability	61
2.5.2.1	Scenario 2D: Variability in initial values of glucose and insulin	63
2.6	Summary	66
3	Output Feedback for Nonlinear SPS using Contraction Analysis	69
3.1	Introduction	69
3.2	System Description and State Feedback	72
3.3	Contractive Controller Design	74
3.3.1	HGO Based Output Feedback and Time Scale Separation	76
3.4	Convergence of Transformed System	77
3.4.1	Peaking and Saturation of Control Law:	84
3.5	Quantitative Bounds for Full Order System	85
3.5.1	Effect of Variation in μ	89
3.6	Simulation Study	92
3.7	Summary	99
4	Incremental Stability of Interconnected Switched Nonlinear Sys- tems	101
4.1	Introduction	101

4.2	Preliminaries and Background	104
4.2.1	Piece-wise Smooth (PWS) Systems	104
4.2.2	Incremental Input-to-State Stability	105
4.3	Interconnected System	110
4.3.1	Interconnected Switched System without Input	112
4.4	Main Result	114
4.4.1	Feedback Interconnection without External Input	114
4.4.2	Cascaded Interconnection without External Input	125
4.5	Simulation Study without Input	130
4.5.1	Case I: Feedback Connection	130
4.5.2	Case II: Cascade Connection	133
4.6	Interconnected System with Input	135
4.6.1	Feedback Interconnection	135
4.7	Simulation Study with Input	142
4.7.1	Case I: Feedback Connection	142
4.7.2	Case II: Cascade Connection	144
4.8	Incremental Stability of Time-Dependant Switched System	147
4.8.1	Average Dwell Time of a Nonlinear Switched System based on Matrix Measure	150
4.8.2	Cascaded Switched System with both Contracting and Non- Contracting Mode	153
4.9	Simulation Study for Time-dependant Switched System	156
4.10	Summary	160
5	Incremental Stability of Systems with Switching Control	163
5.1	Introduction	163
5.2	Problem Formulation	165
5.2.1	Controller Design	166
5.2.2	Unperturbed Dynamics	168
5.2.3	Perturbed Dynamics	169
5.3	Simulation Study	171
5.4	Summary	173
6	Harmonics Suppression in Switched Mechanical Systems	175
6.1	Introduction	175
6.2	Backgraound	177
6.2.1	Problem Formulation	178
6.2.2	System Description	179
6.3	Proposed Solution	181
6.4	Simulation Study	183
6.5	Summary	185

7	Conclusions	187
7.1	Future Scope	191
	References	192
A	Some Results on Partial Contraction	211
B	Notions of Stability	215
B.1	Boundedness and Ultimate Boundedness	215
B.2	Comparison Functions	216
B.3	Input-to-State Stability (ISS)	218
C	Matrix Measure and its Properties	219
C.1	Introduction	219
C.2	Definition of Matrix Measure	219
C.3	Properties of Matrix Measures	220
C.4	Application in Dynamical Systems	222

List of Figures

1.1	Flow of the Thesis	4
1.2	Two trajectories converging to each other in a contracting region.	20
2.1	Estimated plasma glucose concentration for subject 1.	50
2.2	Estimated plasma insulin concentration for subject 1.	51
2.3	Estimated plasma glucose concentration for subject 3.	51
2.4	Estimated plasma insulin concentration for subject 3.	52
2.5	Estimated plasma glucose concentration for subject 5.	52
2.6	Estimated plasma insulin concentration for subject 5.	53
2.7	Plasma glucose concentration trajectories of subject 1 under the proposed observer-based feedback control law.	58
2.8	Plasma glucose concentration trajectories of subject 3 under the proposed observer-based feedback control law.	58
2.9	Plasma glucose concentration trajectories of subject 5 under the proposed observer-based feedback control law.	59
2.10	Insulin infusion rate as suggested by the proposed observer-based feedback control law for subject 1.	59
2.11	Insulin infusion rate as suggested by the proposed observer-based feedback control law for subject 3.	60
2.12	Insulin infusion rate as suggested by the proposed observer-based feedback control law for subject 5.	60
2.13	CVGA for parametric variability of $\pm 30\%$	63
2.14	LBGI for (a) subject 1, (b) subject 3 and (c) subject 5 for Scenarios 2A-2D.	65
2.15	HBGI for (a) subject 1, (b) subject 3 and (c) subject 5 for Scenarios 2A-2D.	66
3.1	Tuning of $(\lambda_{A_o}, \chi_{ao})$	94
3.2	Variation in μ and effects.	95
3.3	Effect of Disturbance on the convergence of $x_3(t)$	97
3.4	Closed-loop trajectory $x_1(t)$ for different ϵ	98
3.5	Tuning closed loop performance with a fixed ϵ	98
4.1	Feedback interconnection of two switched nonlinear systems	111

4.2	Cascade connection of two switched nonlinear systems	112
4.3	Time evolution of variation $\delta\xi$ when $T \in [t/2, t]$	121
4.4	Time evolution of variation $\delta\xi$ when $T \leq t/4 \leq \tau \leq t$	123
4.5	Evolution of variational state $\delta\xi_1$ with respect to time for feedback interconnecton.	132
4.6	Evolution of variational state $\delta\xi_2$ with respect to time feedback interconnecton.	132
4.7	Evolution of variational state $\delta\xi_1$ with respect to time for cascade interconnection.	134
4.8	Evolution of variational state $\delta\xi_2$ with respect to time for cascade interconnection.	134
4.9	Evolution of variational state $\delta\xi_1$ with respect to time for feedback interconnecton.	143
4.10	Evolution of variational state $\delta\xi_2$ with respect to time feedback interconnecton.	144
4.11	Evolution of variational state $\delta\xi_1$ with respect to time for cascade interconnection.	145
4.12	Evolution of variational state $\delta\xi_2$ with respect to time for cascade interconnection.	145
4.13	Evolution of variational state $\delta\xi_1$ when one of the mode is not contracting.	147
4.14	Switching Sequence for x -system (black) and z -system (red)	158
4.15	State x_1 of the cascaded system with respect to time starting from two different initial conditions	159
4.16	State x_2 of the cascaded system with respect to time starting from two different initial conditions	159
4.17	State z_1 of the cascaded system with respect to time starting from two different initial conditions.	160
4.18	State z_2 of the cascaded system with respect to time starting from two different initial conditions.	160
5.1	Temporal evolution of angular position for different initial conditions when subjected to bounded disturbance $\sin(2t)$ without switching control.	172
5.2	Temporal evolution of angular velocity for different initial conditions when subjected to bounded disturbance $\sin(2t)$ without switching control.	172
5.3	Temporal evolution of angular position with switching control.	172
5.4	Temporal evolution of angular velocity with switching control.	173
6.1	Mechanical mass spring damper system with one-sided stiffness.	179
6.2	Coexistence of multiple periodic solutions for a switched mechanical system with one-sided stiffness.	184

6.3	Convergence of trajectories starting from different initial conditions to a steady state solution with a fundamental period same as that of external input.	184
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List of Tables

2.1	Estimated parameters for different subjects [59].	44
2.2	Simulation setting for Scenario 2A-2D for T1D Subjects 1, 3 and 5. . .	60
2.3	Percentage of time spent by the virtual patients in the euglycemic ($x_1 \in [70 - 180]$ mg/dl), hyperglycemic ($x_1 > 180$ mg/dl) and hypo- glycemic ($x_1 \in [50 - 70]$ mg/dl).	62
2.4	Statistical analysis of the closed loop results obtained during Scenario 2A-2D.	65

Abbreviations

ISS	I nter S tate S tability
AP	A rtificial P ancreas
CGM	C ontinuous G lucose M onitoring
IOSS	I nter O utput S tate S tability
HGO	H igh G ain O bserver
SPS	S ingularly P erturbed S ystem
SPP	S ingular P erturbation P arameter
RSS	R educed S low S ystem
SMC	S liding M ode C ontrol
IS	I nsulin S ensitivity
CVGA	C ontrol V ariability G rid A nalysis
EL	E uler L agrange
DT	D well T ime
ADT	A verage D well T ime
UUB	U niformly U ltimately B ounded

Symbols

\mathbb{R}	set of all real numbers
\mathbb{R}_+	set of positive real numbers including zero
$\mathbb{R}_{>0}$	set of positive real numbers
\mathbb{R}^n	real space of dimension n
\mathcal{L}_∞	bounded function space
L	Lipschitz constant
Θ	transformation matrix
J	Jacobian matrix
\exists	there exists
\forall	for all
μ	matrix measure
ϵ	singular perturbation parameter
χ	condition number
id	identity function
D^+	upper Dini derivative
$ a $	absolute value of scalar a
$\ \mathbf{a}\ $	Euclidean norm of vector \mathbf{a}
$a \rightarrow b$	a goes to b
$a \in b$	a belongs to b
$a \triangleq b$	a is defined as b