

**EXISTENCE AND MULTIPLICITY RESULTS FOR
FRACTIONAL AND NONLOCAL ELLIPTIC EQUATIONS**

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EXISTENCE AND MULTIPLICITY RESULTS FOR FRACTIONAL AND NONLOCAL ELLIPTIC EQUATIONS

by

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To My Family

Certificate

This is to certify that the thesis entitled “**Existence and multiplicity results for fractional and nonlocal elliptic equations**” submitted by **Mr. Pawan Kumar Mishra** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

In this thesis, we study the existence and multiplicity of non-negative solutions of non-local elliptic equations with sign-changing nonlinearities. In the first chapter we give the brief survey and preliminary results used in subsequent chapters.

In the second chapter, we study the existence and multiplicity of the solutions of the following problem involving the square root of Laplacian with exponential nonlinearity,

$$(-\Delta)^{\frac{1}{2}}u = h(u), \quad u > 0 \text{ in } (-1, 1), \quad u = 0 \text{ in } \mathbb{R} \setminus (-1, 1),$$

where the nonlinearity $h(u)$ satisfies critical growth of exponential type.

Next we consider the following fractional elliptic equations with sign changing nonlinearity

$$(-\Delta)^{\frac{1}{2}}u = \lambda f(x)|u|^{q-2}u + g(u), \quad u > 0 \text{ in } (-1, 1), \quad u = 0 \text{ in } \mathbb{R} \setminus (-1, 1),$$

where $g(t) = t^p e^{t^\beta}$, $1 < q < 2$, $p > 1$, $1 < \beta \leq 2$, $\lambda > 0$ and the function $f \in L^\gamma((-1, 1))$, for suitable $\gamma > 1$, is sign changing.

Using the Nehari manifold technique we study the multiplicity of solutions. We also study the bifurcation result related to the square root of Laplacian with exponential nonlinearity.

In the third chapter, we study the existence of non-negative solutions for the system of

fractional elliptic equations of the type,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}u &= \frac{p}{p+q}\lambda f(x)|u|^{p-2}u|v|^q + h_1(u,v)e^{u^2+v^2}, \text{ in } (-1,1) \\ (-\Delta)^{\frac{1}{2}}v &= \frac{q}{p+q}\lambda f(x)|u|^p|v|^{q-2}v + h_2(u,v)e^{u^2+v^2}, \text{ in } (-1,1) \\ u, v &\geq 0 \text{ in } (-1,1), u = v = 0 \text{ in } \mathbb{R} \setminus (-1,1), \end{aligned}$$

where $1 < p+q < 2$, $h_1(u,v) = (\alpha+2u^2)|u|^{\alpha-2}u|v|^\beta$, $h_2(u,v) = (\beta+2v^2)|u|^\alpha|v|^{\beta-2}v$ and $\alpha + \beta > 2$. We show the existence of multiple solutions for suitable range of λ by analyzing the fibering maps and the corresponding Nehari manifold. We also study the existence of non-negative solutions for a superlinear system with critical growth exponential nonlinearity.

In the fourth chapter, we study the following Kirchhoff type problem with exponential nonlinearity in the bounded interval $(-1,1)$

$$M\left(\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}}u|^2 dx\right) (-\Delta)^{\frac{1}{2}}u = h(u), u > 0 \text{ in } (-1,1), u = 0 \text{ in } \mathbb{R} \setminus (-1,1),$$

where $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous and h is a real valued function that behaves like e^{u^2} as $u \rightarrow \infty$. We show the existence of mountain-pass solution when the nonlinearity is superlinear near $t = 0$.

Moreover, we have considered the following convex-concave type problem with sign changing nonlinearity

$$M\left(\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}}u|^2 dx\right) (-\Delta)^{\frac{1}{2}}u = \lambda f(x)|u|^{q-2}u + g(u) \text{ in } (-1,1), u = 0 \text{ in } \mathbb{R} \setminus (-1,1),$$

where $f \in L^\gamma((-1,1))$, for some $\gamma > 1$, is sign changing, $1 < q < 2$ and $g(t) = |t|^{p-1}te^{t|t|^\beta}$ for $1 < \beta \leq 2$, $p > 3$ and $\lambda > 0$ is a real parameter.

Also we have considered the Kirchhoff problem involving square root of Laplacian in a bounded domain $\Omega \subset \mathbb{R}^2$ with polynomial critical nonlinearity.

In the fifth chapter, we study the existence of non-trivial solutions for the Kirchhoff p -fractional equation with superlinear critical polynomial nonlinearity using concentration compactness principle due to Lions [72]. In the sequel we studied the following fractional

p -Kirchhoff system

$$\begin{aligned} M\left(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dx dy\right) (-\Delta)_p^s u(x) &= \lambda f(x) |u|^{q-2} u + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta \text{ in } \Omega, \\ M\left(\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}} dx dy\right) (-\Delta)_p^s v(x) &= \mu g(x) |v|^{q-2} v + \frac{2\beta}{\alpha+\beta} |u|^\alpha |v|^{\beta-2} v \text{ in } \Omega, \\ u &= v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{aligned}$$

where $M(t) = a + bt$, $a, b > 0$, $p \geq 2$, $1 < q < p$, $\alpha, \beta \geq 1$, $2p < \alpha + \beta \leq p_s^*$, $ps < n < 2ps$ with $s \in (0, 1)$, λ, μ are real parameters, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and f, g are sign changing weights satisfying some suitable integrability assumptions.

In the sixth chapter, we consider the following polyharmonic Kirchhoff problem:

$$\begin{cases} -M\left(\int_{\Omega} |\nabla^m u|^{\frac{n}{m}} dx\right) \Delta_{\frac{n}{m}}^m u = \frac{f(x,u)}{|x|^\alpha} \text{ in } \Omega, \\ u = \nabla u = \dots \nabla^{m-1} u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $n, m \in \mathbb{N}$ with $n \geq 2m \geq 2$, $0 < \alpha < n$, $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive, continuous function satisfying some growth assumptions and $f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function having critical exponential growth. We also consider the problem involving sign changing weights and using Nehari manifold technique and fibering map analysis, we show the multiplicity of non-negative solutions.

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List of Symbols

Symbol	Meaning
$ A $	Lebesgue measure of a set $A \subset \mathbb{R}^n$.
$\mathcal{B}_r(x)$	Ball of radius r centered at x in $\mathbb{R}^n$.
$C_c^\infty(\Omega)$	Set of infinitely differentiable functions with compact support in Ω .
$\ f\ _p$	Norm of f in $L^p(\Omega)$.
$W^{m, \frac{n}{m}}(\Omega)$	Sobolev space of order m and exponent $\frac{n}{m}$.
$\ u\ _{m, \frac{n}{m}}$	$\left(\int_{\Omega} \nabla^m u ^{\frac{n}{m}} \right)^{\frac{m}{n}}$, norm of u in $W^{m, \frac{n}{m}}(\Omega)$.
$W^{-1, p'}(\Omega)$	Dual space of $W^{1, p}(\Omega)$.
$\ \cdot\ _*$	Norm in $W^{-1, p'}(\Omega)$.
$W_0^{m, \frac{n}{m}}(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $W^{m, \frac{n}{m}}(\Omega)$.
$\ u\ _{W_0^{1, n}}$	$\left(\int_{\Omega} \nabla u ^n dx \right)^{\frac{1}{n}}$, norm of u in $W_0^{1, n}(\Omega)$.
Δ	Laplacian.
$(-\Delta)_p^s$	p -fractional Laplacian, $1 < p < \infty$.
$\mathcal{P}_K$	Non-local operator.
$\ u\ _\infty$	Norm of u in L^∞ space.
$\ u\ _{X_0^p}$	Norm of u in the space X_0^p .
$\ u\ _{E_V}$	Norm of u in the space E_V .
$\ u\ _{W^{\alpha, p}(\Omega)}$	Norm of u in the fractional order spaces $W^{\alpha, p}(\Omega)$.
$L^p(\Omega, w(x)dx)$	Weighted L^p spaces with weight $w(x)$.