Functorial Signal Representation & Base Structured Categories

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by

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submitted in fulfillment of the requirements of the degree of Doctor of Philosophy to the

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To

My Wife
for loving me the way I am and agreeing to walk life's journey with me

My Mother
for being first teacher and never forcing personal ambitious desires on her child

Alexander Grothendieck
for renunciation of wealth, fame, awards, power, family and setting an example for new generation to follow

Beloved Jesus, Koot Hoomi, D.K & Lord Maitreya
for showing the inner path of light when all outer lights went out

Certificate

This is to certify that the thesis titled **Functorial signal representation and Base structured categories**, submitted by **Salil Samant**, to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Place: New Delhi

Declaration

I declare that this dissertation is the result of my own work representing my original intuitions and ideas in my own words and where others ideas or words have been included, I have adequately cited and referenced the original sources to the best of my knowledge. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falisfied any ideas/data/fact/source in my submission.

The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Salil Samant

Date: 27 March 2019

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Finally I would like to thank my family which includes my wife, aai, papa for their constant support.

Salil Samant

Preface

The thesis “Functorial signal representation and Base structured categories” unfolds the intuition underlying compression or in modern terminology sparse nature of a given signal representation by penetrating into the heart of natural generative cause behind the signal.

During my M.tech journey in signal processing almost a decade ago, I was largely influenced in particular by the intuitive visual approach of certain professors especially Prof. Steven M. kay in statistical signal processing theory and Prof. Gilbert Strang in linear algebra. These revered teachers had an extraordinary ability to cut through the ocean of mathematical jargon and make students see what really lies behind the mathematical formulas.

Since then I had a deep desire to understand the structure of information carried by signals (such as a music signal or a painting invoking ecstasy) and always thought that we should be able to represent a signal in the way it was conceived or generated. However not knowing how to proceed on this line of thought I dropped the plan of pursuing this further for the time being. By the end of year 2011 my search while working in industry lead me to the book "Generative theory of shape" which confirmed that I am not a lone ranger among such thinkers and Prof. Michael Leyton was among them conceptualizing a generative cause behind concrete expressions of art, shape, music etc.

While grasping the wreath group approach (utilizing "Visual group theory" by Prof. Nathan Carter) towards the generative structure of many natural processes, I realized that it is the special case of something called ‘Grothendieck Fibration’ in category theory. With this began the earnest study of category theory since mid 2014. I used to present my intuitive understandings of category theory every morning to my guide and we had a lot of fruitful discussion which sparked his interest back into category theory which he first learnt from his mathematics teacher (late) Prof J.B

Srivastava. He was very supportive of my desire to explore this new theory in the quest of my original problem. With my guide I was sure of getting peaceful time to do whatever I wanted to pursue. But of-course with freedom comes responsibility. He used to always encourage me saying "Salil intuition is more important than just precision", whenever I used to feel that I am seriously lacking in mathematical precision which I know I still lack. Moreover as and when I am worried about possibility of erring using pure visual intuitive approach, the following quote by Prof. Grothendieck is the key thing I remind myself often,

"**Discovery is the privilege of the child: the child who has no fear of being once again wrong, of looking like an idiot, of not being serious, of not doing things like everyone else**".

By taking a fresh perspective of category theory on Leyton's model I was able to feel some deeper significance of whole arrow approach. In fact soon I realized that redundancy in signals can only be understood via relative approach of arrow. Of-course once we are able to precisely explain what redundancy itself is, I feel the reason for compression in particular representations will naturally get revealed. This is one central theme explored in the thesis. Moreover the intuitive approach has led to certain new results within category theory itself which I have termed as 'Base structured categories' as well the philosophy of functor acting as a general generative system inherently converting the underlying abstract virtual cause into concrete observable effect in real world as shown in following diagram:

$$\textbf{Cause} \longrightarrow \boxed{\text{Functor}} \longrightarrow \textbf{Effect}$$

In conclusion the thesis is largely influenced by Leyton's generative theory of shape and Grothendieck's relative viewpoint or more generally the arrow perspective of category theory and tackles the fundamental question of what is redundancy and why compression precisely occurs in a representation.

I hope the reader enjoys reading the way I have enjoyed exploring !

Salil Samant

Abstract

The study of categories $(F, \mathbf{C}, \mathbf{D})$ and $(\mathbb{F}, \mathbf{C}, \mathbf{Set})$ as graphs of a functor leads to a proof of these being fibred on $\mathbf{C}$. Grothendieck construction in the context of an ordinary functor $F : \mathbf{C} \to \mathbf{D}$ is examined through the concept of trivial categorification, using an appropriate functor $\mathbf{F} : \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{I} \mathbf{Cat}$ to construct $\int_{\mathbf{C}^{op}} \bar{\mathbf{F}}$. This category characterizes a functor as an abstract right category action while its dual $\mathcal{X} \rtimes_{\mathbf{F}} \mathbf{C}$ or $(\int_{\mathbf{C}^{op}} \bar{\mathbf{F}})^{op}$ characterizes a functor as an abstract left category action. Similarly functor $\mathbb{F} : \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{Set}$ leads to definition of $\mathcal{X} \rtimes_{\mathbb{F}} \mathbf{C}$ and $\int_{\mathbf{C}^{op}} \bar{\mathbb{F}}$ as categories denoting concrete left and right actions of $\mathbf{C}$ respectively. Collectively referred to as 'base structured categories', these are proven abstractly isomorphic to the base category $\mathbf{C}$ but concretely isomorphic to each other or $(\mathbb{F}, \mathbf{C}, \mathbf{Set}) \cong \int_{\mathbf{C}} \bar{\mathbb{F}} \cong \mathcal{X} \rtimes_{\mathbb{F}} \mathbf{C}$. These are special instances of fibred categories where the base category is $\mathbf{C}$ and fibres are $\mathbf{D}$ objects being treated as trivial categories. The perspective of making only base structure explicit through category theory concealing the vertical structure using identity morphisms enables one to combine intuitions of Grothendieck's relative and Leyton's generative theory. As explored further, it greatly facilitates the application of functors in certain fundamental applications such as signal representation which hitherto have been treating objects of category $\mathbf{D}$ such as Hilbert spaces or Riesz spaces purely in a set theoretic way.

Next the foundations of a functorial framework for representing signals are proposed using a functor assisted by base structured categories. By incorporating additional category-theoretic relative and generative perspective alongside the classic set-theoretic measure theory the fundamental concepts of redundancy, compression are formulated in a novel authentic arrow-theoretic way. The existing classic framework representing a signal as a vector of appropriate linear space is shown as a special case of the proposed framework. Further in context of signal-spaces as a categories, various covariant and contravariant forms of L^0 and L^2 functors using categories of measurable or measure spaces and their opposites involving Boolean and measure algebras along with partial extension are studied. We contribute a novel definition of intra-signal redundancy using general concept of isomorphism arrow in a category covering the translation case and others as special cases. Through category-theory we provide a simple yet precise explanation for the well-known heuristic of lossless differential encoding standards yielding better compressions in image types such as line drawings, iconic image, text etc; as compared to classic lossy representation techniques such as JPEG which choose bases or frames in a global Hilbert space.

Finally as an extended exploration of Transformation categories few applications of the base structured categories $\mathcal{X} \rtimes_{\mathbf{F}} \mathbf{C}$, $\int_{\mathbf{C}} \bar{\mathbf{F}}$, $\mathcal{X} \rtimes_{\mathbb{F}} \mathbf{C}$ and $\int_{\mathbf{C}} \bar{\mathbb{F}}$ are proposed and studied. Classic transformation groupoid $X/\!\!/G$ is proven as being a base-structured category $\int_{\mathbf{G}} \bar{F}$. Then using permutation action on a finite set, we introduce the notion of a hierarchy of base structured categories $[(\mathcal{X}_{2a} \rtimes_{\mathbf{F_{2a}}} \mathbf{B}_{2a}) \amalg (\mathcal{X}_{2b} \rtimes_{\mathbf{F_{2b}}} \mathbf{B}_{2b}) \amalg ...] \rtimes_{\mathbf{F_1}} \mathbf{B}_1$ that models local and global structures as a special case of composite Grothendieck fibration. Further utilizing the existing notion of transformation double category $(\mathcal{X}_1 \rtimes_{\mathbf{F_1}} \mathbf{B}_1)/\!\!/\mathbf{2G}$, we demonstrate that a hierarchy of bases naturally leads one from 2-groups to n-category theory. Finally we prove that every classic Klein geometry is the Grothendieck completion ($\mathbf{G} = \mathcal{X} \rtimes_{\mathbb{F}} \mathbf{H}$) of $\mathbb{F} : \mathbf{H} \xrightarrow{F} \mathbf{Man}^{\infty} \xrightarrow{U} \mathbf{Set}$. This is generalized to propose a set-theoretic definition of a groupoid geometry $(\mathcal{G}, \mathcal{B})$ with a principal groupoid $\mathcal{G} = \mathcal{X} \rtimes \mathcal{B}$ and geometry space $\mathcal{X} = \mathcal{G}/\mathcal{B}$; which is essentially same as $\mathbf{G} = \mathcal{X} \rtimes_{\mathbb{F}} \mathbf{B}$ or precisely the completion of $\mathbb{F} : \mathbf{B} \xrightarrow{F} \mathbf{Man}^{\infty} \xrightarrow{U} \mathbf{Set}$.

KEYWORDS: Base structured category, Functor, Grothendieck fibration, Leyton transfer, Functorial signal representation, Redundancy.

सार

फ़ंक्शंस के ग्राफ़ के रूप में श्रेणियों (एफ; सी; डी) और (एफ; सी; सेट) का अध्ययन ए की ओर जाता है एक साधारण के संदर्भ में सी। ग्रोथेंडिक निर्माण पर इनका प्रमाण दिया जा रहा है एफ; सी; डी को तुच्छ श्रेणीकरण की अवधारणा के माध्यम से जांचा जाता है, एक उपयुक्त फ़नकार का उपयोग करके एफ; सी; डी आई ! निर्माण के लिए बिल्ली आर कॉप एफ । इस श्रेणी में एक फंक्शनल को एब्सट्रैक्ट राईट कैटेगिरी एक्शन के रूप में दर्शाती है जबकि इसकी ड्यूल एफ; सी; डी या (आर कॉप एफ) एक अमूर्त वाम श्रेणी कार्रवाई के रूप में एक की विशेषता है। उसी प्रकार एफ; सी; डी ! डी यू ! एफ; सी; डी की परिभाषा पर सेट करें और आर कॉप श्रेणियों के रूप में एफ क्रमशः एफ; सी; डी के बाएँ और दाएँ कार्यों को निरूपित करना। सामूहिक रूप से संदर्भित 'आधार संरचित श्रेणियों' के रूप में, ये आधार के लिए अमूर्त समरूप सिद्ध होते हैं श्रेणी सी लेकिन एक दूसरे या (एफ; सी; सेट) के लिए समवर्ती समरूपता आर सी एफ एक्स ओएफ सी। ये फाइबर्ड श्रेणियों के विशेष उदाहरण हैं जहां आधार श्रेणी सी और फाइबर है डी ऑब्जेक्ट्स को तुच्छ श्रेणियों के रूप में माना जा रहा है। केवल बनाने का परिप्रेक्ष्य आधार संरचना श्रेणी सिद्धांत के माध्यम से स्पष्ट है जो ऊर्ध्वाधर संरचना का उपयोग करके छुपाता है पहचान आकारिकी ग्रोथेंडिक के रिश्तेदार के अंतर्ज्ञान को संयोजित करने में सक्षम बनाती है और लेटन का सामान्य सिद्धांत। जैसा कि आगे पता चला है, यह आवेदन को बहुत सुविधाजनक बनाता है कुछ मूलभूत अनुप्रयोगों जैसे सिग्नल प्रतिनिधित्व में फंक्शनलर्स का उपयोग रिक्त स्थान या जैसे श्रेणी की वस्तुओं का इलाज कर रहे हैं शुद्ध रूप से एक सेट सिद्धांत रूप में। संकेतों का प्रतिनिधित्व करने के लिए एक फंक्शनल फ्रेमवर्क की नींव प्रस्तावित है बेस स्ट्रक्चर्ड कैटेगरीज द्वारा सहायता प्राप्त एक फंक्टर का उपयोग करना अतिरिक्त शामिल करके श्रेणी-सिद्धांत संबंधी सापेक्ष और सामान्य दृष्टिकोण क्लासिक के साथ सेट-थियोरेटिक माप सिद्धांत, अतिरेक, संपीड़न की मूलभूत अवधारणाओं को दर्शाता है एक उपन्यास प्रामाणिक तीर-सिद्धांतिक तरीके से तैयार किए गए हैं। मौजूदा क्लासिक ढांचा उपयुक्त रैखिक स्थान के वेक्टर के रूप में एक संकेत का प्रतिनिधित्व एक विशेष के रूप में दिखाया गया है प्रस्तावित ढांचे का मामला। श्रेणियों के रूप में सिग्नल-स्पेस के संदर्भ में आगे, की श्रेणियों का उपयोग करते हुए और फ़ंक्शनलर्स के विभिन्न सहसंयोजक और रूपों मापनीय या रिक्त स्थान और उनके विपरीत जिनमें बूलियन और माप शामिल हैंआंशिक विस्तार के साथ बीजगणित का अध्ययन किया जाता है। हम एक उपन्यास की परिभाषा में योगदान करते हैं एक श्रेणी में आइसोमोर्फिज्म तीर की सामान्य अवधारणा का उपयोग करके इंट्रा-सिग्नल अतिरेक

अनुवाद के मामले और अन्य को विशेष मामलों के रूप में कवर करना। श्रेणी-सिद्धांत के माध्यम से हम दोषरहित के प्रसिद्ध उत्तराधिकार के लिए एक सरल लेकिन सटीक विवरण प्रदान करते हैं विभेदक एन्कोडिंग मानक छवि प्रकारों में बेहतर संपीड़न जैसे उपज रेखा चित्र, प्रतिष्ठित छवि, पाठ आदि; क्लासिक हानिपूर्ण प्रतिनिधित्व तकनीकों की तुलना में जैसे जेपीईजी जो वैश्विक हिल्बर्ट स्पेस में बेस या फ्रेम चुनते हैं।

अंत में कुछ अनुप्रयोगों में परिवर्तन श्रेणियों की विस्तारित खोज के रूप मेंआधार संरचित श्रेणियों एफ; सी; डी , आरसीएफ, एक्स ओएफ सी औरआरसी प्रस्तावित हैं और का अध्ययन किया। क्लासिक रूपांतरण समूह को आधार-संरचित होने के रूप में सिद्ध किया गया है वर्गआर जी एफ। फिर एक परिमित सेट पर क्रमपरिवर्तन कार्रवाई का उपयोग करके, हम धारणा का परिचय देते हैं आधार संरचित श्रेणियों का एक पदानुक्रम एफ; सी; डी समग्र के एक विशेष मामले के रूप में स्थानीय और वैश्विक संरचनाओं को मॉडल करता है आगे डबल श्रेणी के परिवर्तन की मौजूदा धारणा का उपयोग करना हम प्रदर्शित करते हैं कि आधारों का एक पदानुक्रम स्वाभाविक रूप से एक का नेतृत्व करता है समूह से श्रेणी के सिद्धांत। अंत में हम साबित करते हैं कि हर क्लासिक क्लेन ज्यामिति है ग्रोथेंडीके पूरा होने का मैन यू ! सेट। ये है के साथ एक ज्यामिति की एक सेट-सिद्धांतिक परिभाषा का प्रस्ताव करने के लिए सामान्यीकृत एक प्रमुख समूह और ज्यामिति स्थान; जो अनिवार्य रूप से है या ठीक के पूरा होने पर! मैन यू सेट।

कीवर्ड: बेस स्ट्रक्चर्ड श्रेणी, फ़ंक्टर, ग्रोथेंडिक फ़िब्रेशन, लेटन ट्रांसफर, फंक्टोरियल सिग्नल प्रतिनिधित्व, अतिरेक।

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Notation and terminology

In thesis, the upright Roman boldface font such as $\mathbf{C}$ or **Riesz** denote categories. In particular, if we want to emphasize the concrete nature of a particular category the notation $(\mathbf{C}, U)$ is used. Functors in general are denoted by capital letters such as F or L^2. Particularly, if we want to emphasize the concrete nature of a functor we use notation $F : (\mathbf{C}, U) \to (\mathbf{D}, V)$ utilizing concrete notations for categories. In particular an upright bold letter in case of a functor such as $\mathbf{F} : \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{I} \mathbf{Cat}$, specially denotes that it is a composite functor (post composing general F with trivial categorifying functor I) involving new concept of trivial categorification. We use blackboard bold (double-struck) letters for composite functor (post composing general F with underlying functor or concrete functor $U : (\mathbf{D}, U) \to (\mathbf{Set}, ID)$) $\mathbb{F} : \mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{U} \mathbf{Set}$ where U is the underlying faithful functor of the concrete category $(\mathbf{D}, U)$ over $\mathbf{Set}$. We use 'pseudo-functor on $\mathbf{B}$' to refer to a contravariant pseudo-functor $\Psi : \mathbf{B} \to \mathbf{Cat}$. This is thought of as a covariant pseudo-functor $\bar{\Psi} : \mathbf{B}^{op} \to \mathbf{Cat}$. It is often notation-wise abused and denoted simply as $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$. Then the Grothendieck construction (original version) of this contravariant pseudo-functor Ψ, is denoted as $\int_{\mathbf{B}} \Psi$.

$(F, \mathbf{C}, \mathbf{D})$	Abstract graph of a functor
$(\mathbb{F}, \mathbf{C}, \mathbf{Set})$	Concrete graph of a functor
$\int_{\mathbf{C}^{op}} \bar{\mathbf{F}}$	Abstract right action of $\mathbf{C}$ on objects in $F\mathbf{C}$ as trivial categories
$\mathcal{X} \rtimes_{\mathbf{F}} \mathbf{C}$	Abstract left action of $\mathbf{C}$ on objects in $F\mathbf{C}$ as trivial categories
$\mathcal{X} \rtimes_{\mathbb{F}} \mathbf{C}$	Concrete left action of $\mathbf{C}$ on underlying sets of $F\mathbf{C}$
$\int_{\mathbf{C}^{op}} \bar{\mathbb{F}}$	Concrete right action of $\mathbf{C}$ on underlying sets of $F\mathbf{C}$
$\mathbf{D}^{\rightarrow}$	Arrow category of category $\mathbf{D}$
$f \restriction I$	Restriction $(f\vert_I)$ of function f to domain I
Meas	Category of measurable spaces (X, Σ_X) and measurable maps (functions) $f : X \to Y$
Measure	Category of measure spaces (X, Σ_X, μ) and inverse-measure-preserving morphisms $f : X \to Y$
LocMeas	Category of localizable measurable spaces $(X, \Sigma_X, \mathcal{N}(\mu))$ and $f^\bullet$ a.e. equivalence classes of non-singular measurable maps $f : X \to Y$
LocMeasure	Category of localizable measure spaces (X, Σ_X, μ) and $f^\bullet$ a.e. equivalence classes of inverse-measure-preserving maps $f : X \to Y$

compBoolAlg	Category of Dedekind σ-complete Boolean algebras $\mathfrak{B}$ and sequentially order-continuous Boolean homomorphisms $\pi : \mathfrak{B} \to \mathfrak{A}$
MeasureAlg	Category of measure algebras $(\mathfrak{B}, \bar{\mu})$ and SOC measure-preserving Boolean homomorphisms $\pi : \mathfrak{B} \to \mathfrak{A}$
countMeas	Subcategory of **Meas** (or **LocMeas**) of measurable spaces $(X, \mathcal{P}X)$ (or $(X, \mathcal{P}X, \phi)$)and measurable maps (functions) $f : X \to Y$
countMeasure	Subcategory of **Measure** (or **LocMeasure**) of measure spaces $(X, \mathcal{P}X, count)$ and inverse-measure-preserving maps $f : X \to Y$
Riesz	Category of Riesz spaces and Riesz homomorphisms
BanLatt	Category of Banach lattices and bounded Riesz homomorphisms
Hilb	Category of Hilbert spaces and continuous(bounded) linear maps
compBoolAlg	Category of Dedekind σ-complete Boolean algebras $\mathfrak{B}$ and sequentially order-continuous Boolean homomorphisms $\pi : \mathfrak{B} \to \mathfrak{A}$
$\mathfrak{L}^0(X, \Sigma_X)$	(or $\mathfrak{L}^0 = \mathfrak{L}^0(\mu)$) Space of virtually measurable real-valued functions f defined on conegligible subsets of X
$\mathfrak{L}^0_X$	Space of measurable real-valued functions f defined on X
$\mathfrak{L}^p(X, \Sigma_X)$	(or $\mathfrak{L}^p = \mathfrak{L}^p(\mu)$, $p \in (1, \infty)$) Set of functions $f \in \mathfrak{L}^0 = \mathfrak{L}^0(\mu)$ such that $\lvert f\rvert^p$ is integrable
$L^0(X, \Sigma_X, \mu)$	(or $L^0 = L^0(\mu) = L^0(X, \Sigma_X, \mathcal{N}(\mu))$) Set of equivalence classes $f^\bullet$ in $\mathfrak{L}^0(\mu)$ under $=_{\text{a.e.}}$
$L^p(X, \Sigma_X, \mu)$	(or $L^p = L^p(\mu)$, $p \in (1, \infty)$) Set of functions $\{f^\bullet : f \in \mathfrak{L}^p\} \subseteq L^0 = L^0(\mu)$ in $\mathfrak{L}^0(\mu)$ under $=_{\text{a.e.}}$