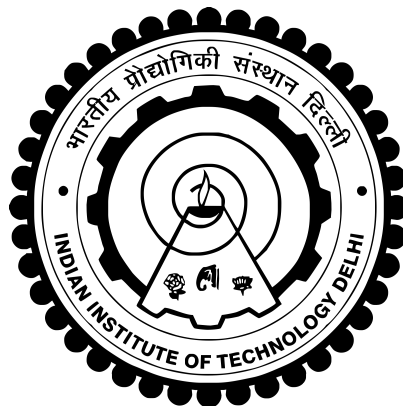


SOME STUDIES ON RAMANUJAN SUMS

DEVENDRA KUMAR YADAV



DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAY 2019

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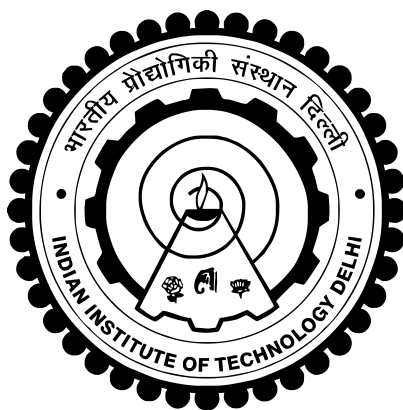
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DEPARTMENT OF ELECTRICAL ENGINEERING

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAY 2019

Dedicated to

My Family

&

Friends

Certificate

This is to certify that the thesis titled **Some studies on Ramanujan Sums** being submitted by **Devendra Kumar Yadav** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** is the record of the bona fide research work carried out by him under my supervision. In my opinion, the thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted either in part or in full to any other university or institute for the award of any degree or diploma.

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Devendra Kumar Yadav

Abstract

In 1918, about a century ago, S. Ramanujan defined sum of certain trigonometrical sums, which are now being called as Ramanujan Sums. The primary objective of defining these sums, by then, was to represent various arithmetical functions in the form of Ramanujan Sums. These sums have been extensively used in the field of number theory. Recently, there is renewed interest in these sums in the field of signal processing, particularly in period extraction. The main objective of this thesis is twofold: first, to explore Ramanujan Sums in other areas of signal processing; second, to develop mathematical framework for Discrete Wavelet Transform, for finite duration signals, by considering a novel family of sequences as a basis.

In this thesis, we define two families of operators, based on Ramanujan Sums, termed as Ramanujan *class* – 1 operators and Ramanujan *class* – 2 operators. The *class* – 1 family of Ramanujan operators is defined by considering one period of Ramanujan Sum as its kernel having the property of LTI system. It is further proved that these *class* – 1 operators satisfy the properties of first derivative in the digital domain. Further, considering one period of Ramanujan Sums with a particular shift and taking this shifted sequence as a kernel of an operator, it is shown that these operators satisfy the properties associated with second derivatives in the digital domain, which are termed as Ramanujan *class* – 2 operators. After proving that these operators satisfy the properties of derivatives, their efficacy in applications where detail part of the signal is used to extract the information from the signal, is established. Two of them are edge detection and noise level estimation. A relationship between Ramanujan Sums and various edge detection masks is also investigated. It is also shown that various masks which exist in literature such as Robert's, Prewitt's and others are particular case of masks generated by Ramanujan Sums.

Going further, a new family of sequences, namely, Orthogonal Ramanujan Sums (ORS) is defined based on Ramanujan Sums. A new representation of any finite length or periodic signal is given, namely, Orthogonal Ramanujan Periodic Transform (ORPT), using ORS. A novel family of Multirate Filter Banks, based on divisors of signal length, using Orthogonal Ramanujan Sums, has also been proposed. It has been shown that ORS can also be used to achieve a novel $q - adic$ multiresolution analysis. A $q - adic$ multiresolution approach means the scaling and translation parameter in the wavelet function can be any positive integer q , and not necessarily equal to 2, as is often used. This approach shows that instead of using same translates and dilates for expansion, one can use different translates and dilates at different scales for expanding elements of $l^2(\mathbb{Z}_{\mathbb{N}})$. Orthogonal Ramanujan Sums provide one possible approach to achieve this. For $q - adic$ multiresolution analysis, a signal needs to be decomposed into q -bands. ORS divides a signal into q orthogonal bands. Using this approach a signal can be decomposed in different number of bands at different resolutions. This gives the flexibility to analyse signal with varying resolutions. A complete theory using ORS has also been developed for multidimensional signal.

As a spin-off arising from the work, two new interesting properties of Ramanujan Sums are also found. One of these property shows that Ramanujan Sums corresponding to the powers of prime are upsampled version of lower order Ramanujan sums, of the same prime with some multiplication factor. Second property shows that the product of two arbitrary but fixed shifted Ramanujan Sums also forms a shifted Ramanujan Sum, with some delay. A closed-form expression for this delay is also derived in this thesis.

सार

1918 में, लगभग एक सदी पहले, एस रामानुजन ने कुछ त्रिकोणमितीय योगों का योग किया था, जिसे अब रामानुजन सम कहा जाता है। तब तक, इन योगों को परिभाषित करने का प्राथमिक उद्देश्य, विभिन्न अंकगणितीय कार्यों का रामानुजनम सम के रूप में प्रतिनिधित्व करना था। संख्या सिद्धांत में इन योगों का बड़े पैमाने पर उपयोग किया गया है। हाल ही में, सिग्नल प्रोसेसिंग में विशेष रूप से अवधि निष्कर्षण में इन योगों में नए सिरे से रुचि है। इस थीसिस का मुख्य उद्देश्य दो गुना है: प्रथम, सिग्नल प्रोसेसिंग के अन्य क्षेत्रों में रामानुजन समों का पता लगाने के लिए; दूसरा, एक आधार के रूप में एक नवपरिवार के परिणामों पर विचार करके, डिस्क्रेट वेवलेट ट्रांसफॉर्म के लिए गणितीय रूपरेखा विकसित करना।

इस थीसिस में, हमने रामानुजन सम पर आधारित ऑपरेटर के दो परिवारों को परिभाषित किया है, रामानुजन वर्ग - 1 ऑपरेटर और रामानुजन वर्ग- 2 ऑपरेटर। वर्ग - 1 परिवार रामानुजन ऑपरेटर रामानुजन सम के एक काल को एलटीआई प्रणाली की संपत्ति वाले कर्नेल द्वारा परिभाषित किया गया है। यह साबित हुआ है कि ये वर्ग - 1 ऑपरेटर डिजिटल डोमेन में पहले व्युत्पन्न के गुणों को संतुष्ट करते हैं। आगे, विचार कर एक विशेष बदलाव के साथ रामानुजन सम की एक अवधि और इस स्थानांतरित अनुक्रम के रूप में किसी ऑपरेटर का कर्नेल, यह दिखाया गया है कि ये ऑपरेटर डिजिटल डोमेन में दूसरे व्युत्पन्न के साथ जुड़े हुए गुणों को पूरा करते हैं, जिसे रामानुजन वर्ग - 2 ऑपरेटर कहा जाता है। यह साबित करने के बाद कि ये ऑपरेटर व्युत्पन्न के गुणों को संतुष्ट करते हैं, उनके अनुप्रयोगों में प्रभावकारिता, जहां सिग्नल का विस्तार हिस्सा संकेत से जानकारी निकालने के लिए उपयोग किया जाता है, स्थापित किया गया है। उनमें से दो किनारे का पता लगाने और शोर स्तर के आकलन हैं। रामानुजन सम और विभिन्न एज डिटेक्शन मास्क के बीच एक संबंध की जांच भी की है। यह भी दिखाया गया है कि विभिन्न मुखौटे जो साहित्य में मौजूद हैं जैसे कि रॉबर्ट, प्रिविट और अन्य रामानुजन सम द्वारा उत्पन्न मास्क के विशेष मामले हैं।

आगे जाकर, अनुक्रमों का एक नया परिवार, ऑर्थोगोनल रामानुजन सम (ओआरएस) है जो कि रामानुजन सम पर आधारित है। ओआरएस का उपयोग करके किसी भी परिमित लंबाई या आवधिक का एक नया प्रतिनिधित्व संकेत दिया है, जिसे ऑर्थोगोनल रामानुजन आवधिक ट्रांसफॉर्म (ओआरपीटी) नाम दिया है। ऑर्थोगोनल रामानुजन सम का उपयोग करते हुए सिग्नल लंबाई के विभाजकों पर आधारित मल्टीरेट फ़िल्टर बैंकों का एक नवपरिवार भी प्रस्तावित किया गया है। यह दिखाया गया है कि ओआरएस का इस्तेमाल क्यू - एडिक विश्लेषण के लिए भी हो सकता है। क्यू - एडिक मल्टी-रिजॉल्यूशन का मतलब है कि तरंगिका फ़ंक्शन में स्केलिंग और अनुवाद पैरामीटर कोई भी सकारात्मक पूर्णांक q हो सकता है, और जरूरी नहीं कि 2 के बराबर, जैसा कि अक्सर उपयोग किया जाता है। यह दृष्टिकोण दिखाता है कि विस्तार के लिए एक ही अनुवाद और स्केलिंग का उपयोग करने के बजाय $\text{el}2(\text{ZN})$ के तत्वों के विस्तार के लिए विभिन्न पैमानों पर एक अलग का उपयोग कर सकते हैं। ऑर्थोगोनल रामानुजन सम इसे प्राप्त करने के लिए एक संभव दृष्टिकोण प्रदान करता है। क्यू - एडिक मल्टी-रिजॉल्यूशन विश्लेषण के लिए, एक संकेत को क्यू-बैंड में विघटित करने की आवश्यकता होती है। ओआरएस एक संकेत को क्यू ऑर्थोगोनल बैंड में विभाजित करता है। इस दृष्टिकोण का उपयोग करके एक सिग्नल विभिन्न प्रस्तावों पर बैंड को अलग-अलग संख्या में विघटित

किया जा सकता है। यह अलग-अलग के साथ सिग्नल का विश्लेषण करने के लिए लचीलापन भी देता है। ओआरएस का उपयोग करने वाला एक पूरा सिद्धांत भी बहुआयामी संकेत के लिए विकसित किया गया है।

काम से उत्पन्न होने वाले स्पिन-ऑफ के रूप में, रामानुजन सम के दो नए दिलचस्प गुण भी पाए हैं इनमें से एक गुण से पता चलता है कि अभाज्य घात के लिए रामानुजन सम इससे संबंधित निचले क्रम की शक्तियां के रामानुजन सम के वर्जन हैं । दूसरी गुण से पता चलता है कि दो रामानुजन सम का उत्पाद मनमाने लेकिन स्थाई रूप से स्थानांतरित रामानुजन सम भी एक स्थानांतरित रामानुजन योग बनाते हैं लेकिन कुछ देरी के साथ । इस देरी के लिए एक बंद रूप अभिव्यक्ति भी इस थीसिस में व्युत्पन्न है।

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List of Notations and Abbreviations

$\exists$	there exists
$a b$	a divides b
$a \nmid b$	a does not divides b
(a, b)	greatest common divisor of a and b
$(a, b) = 1$	a and b are relatively prime
$\phi(n)$	number of positive integers not greater than and prime to n
$\mathbb{Z}$	Set of integers
$\mathbb{Z}^+$	Set of positive integers
$\mathbb{R}$	Set of Real numbers
FT	Fourier Transform
STFT	Short-Time Fourier Transform
CWT	Continuous Wavelet Transform
DFT	Discrete Fourier Transform
IDFT	Inverse Discrete Fourier Transform
DSP	Digital Signal Processing
DWT	Discrete Wavelet Transform
FIR	Finite Impulse Response
MRA	Multiresolution Analysis

RFB	Ramanujan Filter Bank
RPT	Ramanujan Periodic Transform
RFT	Ramanujan FIR Transform
RSE	Ramanujan Sum Expansion
ORS	Orthogonal Ramanujan Sums
ORPT	Orthogonal Ramanujan Periodic Transform