ON MODELS OF IRREDUCIBLE REPRESENTATIONS OF CERTAIN LIE ALGEBRAS AND SPECIAL FUNCTIONS

by

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CERTIFICATE

This is to certify that the thesis entitled "ON MODELS OF IRREDUCIBLE REPRESENTATIONS OF CERTAIN LIE ALGEBRAS AND SPECIAL FUNCTIONS", which is being submitted by Mr. Vivek Sahai for the award of the degree, DOCTOR OF PHILOSOPHY, to the Indian Institute of Technology, Delhi, is a bonafide record of research work done under my guidance and supervision.

The thesis has reached the stage of fulfilment of the requirements and regulations related to the degree. The results obtained in this thesis have not been submitted to any other Institute or University for the award of any degree or diploma.

H. L. Manocha
Supervisor.

IN MEMORY OF

MY GRANDFATHER

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SYNOPSIS

Lie theory - the theory of Lie groups, Lie algebras and their applications - is a fundamental part of mathematics. It has been the focus of burgeoning research effort, and is now seen to encompass a tremendous spectrum of mathematical areas. Research works of Lie, Cartan, Killing and Weyl have led to its recent most elegant form which is extremely useful in many applications of mathematics. Indeed, the applications of Lie theory are astonishing in their pervasiveness and sometimes in their expectedness.

The past four decades have witnessed mathematicians showing keen interest in relating Lie theory to hypergeometric functions of one and more variables. Though, to some, expressions 'Lie theory and special functions' sound like 'East and West', the works of Weisner, Vilenkin, Miller, Kalnins, Manocha and a few others have eloquently demonstrated that there is a natural and deep connection between Lie theory and special functions. The scenario at present is that Lie theory has tremendously enriched the theory of special functions, while special functions have repaid it by bringing 'concreteness' in the 'abstractness' of Lie theory. Indeed, the two are moving hand in hand with prospects of further strengthening of ties between the two. One aspect of Lie theory which has made a tremendous contribution to this development is the representation theory of Lie groups and Lie algebras. To be precise, the connection between Lie theory and special functions has been by and large based on the following sequence :

1. Construction of a model of representation, preferably irreducible, of a Lie algebra on the representation space involving special functions.

2. Exponention of the above model so as to have a model of representation of the corresponding Lie group.

3. Exploitation of the model in (2) for obtaining identities involving special function appearing in (1).

As can be seen, the most important step in the above sequence is constructing a model of representation of a Lie algebra with special functions appearing in the representation space. To accomplish this, there is a need for developing theory on this. Indeed, theory on this has been developed by Weisner, Vilenkin, Miller, Kalnins and Manocha, to quote a few. However, there is a need for enriching and growing this theory further. It is precisely this role of developing theory and constructing models which the proposed thesis takes upon itself.

The thesis will consist of six chapters. A brief insight of each chapter is given below :

Chapter 1. **Introduction**

This chapter is introductory in nature and provides all the basic information needed for the subsequent chapters. It mainly contains two sections. The first section deals with Lie groups and Lie algebras, while the second one is devoted to special functions. An effort is made to make the thesis self contained as far as possible.

Chapter 2. **Models based on fractional differentiation**

A theorem is established which suggests various irreducible representations of the special linear algebra $sl(2,\mathbb{C})$. Guided by

this theorem, models of representations are constructed, with special functions ${}_2F_1$ and ${}_1F_1$ appearing in the representation space. Sensing that these models have the potential to induce new models, theory of fractional calculus is introduced and then invoked. This helps in constructing new models, with hypergeometric functions ${}_{k+2}F_{k+1}$ and ${}_{k+1}F_{k+1}$ appearing in the representation spaces. Later, the new models are fully exploited for obtaining identities, believed to be new.

Chapter 3. **Models based on Mellin transformation**

Two theorems are reproduced which provide guidelines for constructing models of irreducible representations of the Lie algebras $sl(2,\mathbb{C})$ and oscillator algebra $\mathcal{G}(0,1)$. Based on these, models of representations are constructed, with the Laguerre polynomials $L_m^{(\alpha)}(x)$ appearing in the representation space. To translate these models into new ones, Mellin transformation is introduced. This helps in constructing new models in terms of difference differential operators, with the polynomials $S_m^{(\alpha)}(\beta;X)$, defined in terms of $F_D^{(n)}$, serving as the representation space. Through exponentiation, these models lead to new interesting results and identities.

Chapter 4. **Models based on Euler transformation**

Based on a theorem due to Miller, models of irreducible representations of $sl(2,\mathbb{C})$ and $\mathcal{G}(0,1)$ in one variable are introduced. Through another theorem, which is established, these models are upgraded into those involving two variables. A transformation based on Euler integral is introduced, which transforms the above models into deeper ones with hypergeometric functions appearing in the representation space. This exercise

leads to recurrence relations in terms of difference operators.

Chapter 5. **Models of q-representations**

Taking a cue from the work of Manocha, theory involving q-representations is developed. Based on this theory, models of irreducible q-representations of $\mathcal{G}_q(0,0)$, $\mathcal{G}_q(0,1)$ and $\mathcal{G}_q(1,0)$ are constructed. As $q \longrightarrow 1$ all these models reduce to those of new ones.

Chapter 6. **Models of irreducible unitary representation of compact Lie group SU(2)**

The compact Lie group $SU(2)$ and its real Lie algebra $su(2)$ are introduced. An attempt is made to construct a model of an irreducible unitary representation of $SU(2)$ on a finite dimensional Hilbert space involving hypergeometric functions ${}_2F_1$. To accomplish this, firstly a model of a finite-dimensional irreducible representation of $sl(2,\mathbb{C})$ in one variable is constructed. A theorem is established which upgrades this model to a two variable model. Since $su(2)$ is a real part of $sl(2,\mathbb{C})$, meaning that $sl(2,\mathbb{C})$ is the complexification of $su(2)$, restriction of this model to $su(2)$ gives a finite-dimensional irreducible representation of $su(2)$ on a Hilbert space. Through a similarity transformation, this model is transformed into the one involving hypergeometric functions ${}_2F_1$. By carrying out exponentiation, the required model of irreducible unitary representation of $SU(2)$ is established. This model, in turn, gives rise to interesting identities.

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