RANDOM SAMPLING IN REPRODUCING KERNEL SUBSPACES

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by

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Dedicated to

My Brothers and Sister

Certificate

This is to certify that the thesis entitled **"Random sampling in reproducing kernel subspaces"** submitted by **Mr. Dhiraj Patel** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by his under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Date:
Place: New Delhi

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"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment." *–Carl Friedrich Gauss*

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New Delhi *Dhiraj Patel*

Abstract

This thesis is concerned with studying the random sampling problem in the reproducing kernel function spaces categorized into three types.

In the first framework, the function space is the image space of an idempotent integral operator on Lebesgue space, mixed Lebesgue space, and the Orlicz space under certain off-diagonal decay and regularity condition. The function space includes shift-invariant space, the space of functions with a finite rate of innovation. For the case of Lebesgue space, the repeated application of Bernstein's inequality for the sum of independent random variables, we examine the stability of uniformly distributed random sample points over a bounded cube with an overwhelming probability and analyze the order of the sample size. For mixed Lebesgue space and Orlicz space, the probability is estimated from the covering radius of the uniformly distributed random samples.

In the second setting, we consider the localized reproducing kernel subspace of $L^p(\mathbb{R}^n)$, which includes the function space in the first model, the quasi-shift invariant space, and the weighted Fock space. The sampling discretization of the integral norm for the finite-dimensional space gives the probability bound for the stability of uniformly distributed random samples over compact set Ω for the set of concentrated functions on Ω.

In the third case, the function space is an infinite-dimensional reproducing kernel Hilbert space $\mathcal{H}$ of square integral functions on the unit sphere $\mathbb{S}^{n-1}$. With an overwhelming probability, the uniformly distributed random samples over $\mathbb{S}^{n-1}$ is a stable sample set for a class of localized functions in $L^2(\mathbb{S}^{n-1})$. The probability bound of the random sampling inequality for the localized function is obtained from the matrix Bernstein inequality on the independent random matrices.

The reconstruction scheme to reconstruct the concentrated function from its random samples is proposed in this thesis. We first propose the local reconstruction formula for the reconstruction of the functions in a finite-dimensional space from their random samples. As a consequence, the concentrated function can be locally recovered from their random sample values. We also provide an exponential converges iterative reconstruction algorithm for the global reconstruction of the concentrated functions and illustrate the proposed scheme with an example.

सार

यह थीसिस तीन प्रकारों के पुनरुत्थान करने वाले कर्नल द्वारा उत्पन्न फलन स्पेस में यादृच्छिक नमूनाकरण समस्या का अध्ययन करने से संबंधित है।

पहले मामले में, फलन स्पेस लेबेग स्पेस, मिश्रीत लेबेग स्पेस, और ऑरलिज़ स्पेस के ऊपर एक समाकलन आईडेम्पोटेंट ऑपरेटर के परास स्पेस होता है, जहां समाकलन कर्नेल कुछ ऑफ–दिअगोंल क्षय और नियमितता की शर्तें मानते है। माने गए फलन स्पेस में शिफ्ट–इन्वारिंत स्पेस और सिमित दर इनोवेशन फलन स्पेस शामिल है। लेबेग स्पेस के मामले में, स्वतंत्र यादृच्छिक चर की योग पे लगातार बर्नस्टीन के असमानता का अनुप्रयोग करके, हम अत्यधिक संभाव्यता के साथ सिमित घन्क्षेत्र में समान रूप से वितरित यादृच्छिक नमूना बिंदुओं की स्थिरता की जांच करते हैं और नमूना आकार के क्रम का विश्लेषण करते हैं। मिस्त्रित लेबेग स्पेस और ऑरलिज़ स्पेस के लिए स्थिरता की संभावना समान रूप से वितरित यादृच्छिक नमूने के कवरिंग त्रिज्या से अनुमानित करते है।

दूसरी सेटिंग में, हम $L^p(\mathbb{R}^n)$ के स्थानीय पुनरुत्पादन कर्नेल उपस्थान पर विचार करते हैं, जिसमें पहले मॉडल स्पेस के साथ लगवाग शिफ्ट–इन्वारिंत स्पेस और भारित फोक्क स्पेस शामिल होते है। परिमित–आयामी स्थान के लिए समाकलित मानदंड का नमूना विवेकीकरण द्वारा संहतसमुच्चय पर केन्द्रित फलन समुच्चय के लिए यादृच्छिक नमूनों की स्थिरता की संभावना देता है जो संहतसमुच्चय पर समान रूप से वितरित है।

तीसरी परिस्थिति में, फलन स्पेस एक अनंत–आयामी पुनरुत्थान कर्नेल हिल्बर्ट स्पेस $\mathcal{H}$ है जो एकक गोलाकार क्षेत्र $\mathbb{S}^{n-1}$ पर बर्ग समाकलनीय फलन का है। भारी संभावना के साथ, $\mathbb{S}^{n-1}$ पर समान रूप से वितरित यादृच्छिक नमूने $L^2(\mathbb{S}^{n-1})$ में स्थानीयकृत फलन के वर्ग के लिए एक स्थिर नमूना समुच्चय है। स्थानीय फलन के लिए यादृच्छिक नमूनाकरण असमानता की संभाव्यता स्वतंत्र यादृच्छिक आव्यूह पर आव्यूह बर्नस्टीन असमानता से प्राप्त की जाती है।

इस थीसिस में यादृच्छिक नमूनों से केंद्रित कार्य के पुनर्निर्माण के लिए पुनर्निर्माण योजना प्रस्तावित है। हम पहले परिमित–आयामी स्पेस में यादृच्छिक नमूनों से फलन के पुनर्निर्माण के लिए स्थानीय पुनर्निर्माण सूत्र का प्रस्ताव करते हैं। परिणामस्वरूप, केंद्रित फलन को उनके यादृच्छिक नमूना मूल्यों से स्थानिक पुनर्प्राप्त किया जा सकता है। हम केंद्रित फलन के वैश्विक पुनर्निर्माण के लिए एक चरघातांकी अभिसृत पुनरावृत्त पुनर्निर्माण एल्गोरिथ्म भी प्रदान करते हैं और प्रस्तावित योजना को एक उदाहरण के साथ चित्रित करते हैं।

Contents

List of Figures

List of Tables

List of Symbols

Symbol	**Meaning**
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{Z}^n$	The set of n-dimensional integer space.
$\mathbb{R}^n$	The set of n-dimensional Euclidean space.
$\mathbb{S}^{n-1}$	The set of unit sphere in $\mathbb{R}^n$.
σ_n	Surface measure on $\mathbb{S}^{n-1}$.
$\|a\|_p$	ℓ^p-norm on the finite array a.
p'	$\frac{1}{p} + \frac{1}{p'} = 1$, for $1 \leq p \leq \infty$.
$\partial\Omega$	Boundary of the set Ω.
$\Omega_1 \times \Omega_2$	Cartesian product of the sets Ω_1 and Ω_2.
C_R	$\left[-\frac{R}{2}, \frac{R}{2} \right]^n$
$C_{R,S}$	$\left[-\frac{R}{2}, \frac{R}{2} \right]^n \times \left[-\frac{S}{2}, \frac{S}{2} \right]$
$\mathbb{E}(X)$	Expectation of random variable X.
$Var(X)$	Variance of random variable X.
$\mathbb{P}(X)$	Probability of the event X.
μ	Lebesgue measure.
χ_A	The characteristic function on the set A.
$B(x;r)$	r radius open ball centered at x with respect to $\|\cdot\|_\infty$.
$\#S$	Number of elements in the finite set S.
$N_0(\Gamma)$	Supremum number of elements in Γ in the unit cube, i.e., $\sup_{k \in \mathbb{Z}^n} \#\Big(k + \left[-\frac{1}{2}, \frac{1}{2} \right]^n \cap \Gamma\Big)$.
$\delta_{\gamma,\gamma'}$	Kronecker delta.
Id	Identity operator.

Abbreviation	Meaning
RKHS	reproducing kernel Hilbert space.
RKBS	reproducing kernel Banach space.
i.i.d.	independent identically distributed.
BUPU	bounded uniform partition of unity.