WEIGHT OF A GROUP AND UNIT GROUP OF GROUP RINGS

By

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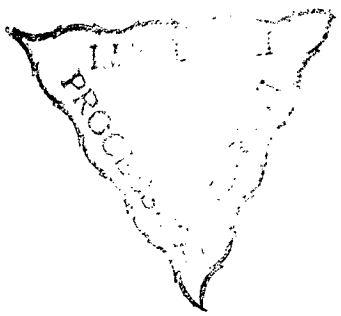
Group theory



To My Baba

Certificate

We are satisfied that the thesis entitled *Weight of a Group and Unit Group of Group Rings* submitted by *Ms Megha Gupta*(2003MAZ0001) is worthy of consideration for the award of the degree of Doctor of Philosophy and is a record of the original bona fide research work carried out by her under our guidance and supervision. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

R. K. Sharma.

Prof. R. K. Sharma **Prof. J. B. Srivastava**

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Abstract

A group G is said to have finite weight if G is normally generated by a finite subset, i.e., $G = \langle X \rangle_N$ = the normal closure of X in G, for some finite subset X of G. The weight $w(G) = n < \infty$ if n is the least positive integer such that $G = \langle X \rangle_N$ where X has n distinct elements. P. Kutzko (1976) proved that *if G is a group of finite weight and the lattice of normal subgroups of G contained in the commutator subgroup G' satisfies the minimum condition then* $w(G) = w(G/G')$. Thus $w(G) = w(G/G')$ for any finite group G. A. H. Rhemtulla (1981) carried this study further. It follows that if G is a solvable group of finite weight then $w(G) = w(G/G')$.

We have studied groups for which conditions of finite weight and finitely generated are equivalent. More classes of groups have been found out for which $w(G) = w(G/G')$. We have proved that nilpotent groups of finite weight are finitely generated. Further it is shown that nilpotent groups of weight 1 are cyclic. An example of a solvable group of finite weight is constructed which is not finitely generated. For an FC group G of finite weight it is shown that $w(G) = w(G/G')$ and also that G is finitely generated. For solvable by FC groups of finite weight and for finitely generated FC-nilpotent groups it is proved that $w(G) = w(G/G')$. Further, for any group A and a non torsion group B of finite weight such that the set of all torsion

elements of B form a subgroup, it is shown that in case of unrestricted wreath product $w(A\ Wr\ B) = w(B)$. The weight of linear groups $GL_n(K)$, $UT_n(K)$ and $T_n(K)$, for any field K have been computed. Also weight of all p-groups of order p^n, $n \leq 5$, p an odd prime are obtained. The weight of all groups of order up to 100 have been tabulated.

For the dihedral group D_{2n} of order $2n$, we completely characterize the bicyclic units in the integral group ring $\mathbb{Z}D_{2n}$ for all n. We prove that the index of the subgroup generated by all bicyclic units in the unit group $\mathcal{U}(\mathbb{Z}D_{2n})$ is infinite except when $n = 3, 4$ and 6. We compute this index when $n = 3, 4$ and 6. We determine the number of distinct Bass cyclic units up to inverses in $\mathbb{Z}D_{2p}$ and $\mathbb{Z}D_{4p}$, for any odd prime p. We aim to give a characterization of $\mathcal{U}_1(\mathbb{Z}D_{10})$ that highlights the role of bicyclic units. Further we study the relationship between weight of certain groups and the unit groups of integral group ring.

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