ON EXTREMAL PROPERTIES AND LOCATION OF ZEROS OF POLYNOMIALS

Brham Datt

Thesis submitted to the Indian Institute of Technology, Delhi
for the award of the degree of
Doctor of Philosophy

Department of Mathematics
Indian Institute of Technology, Delhi

January 1976

CERTIFICATE

This is to certify that the thesis entitled "On Extremal Properties and Location of Zeros of Polynomials" which is being submitted by Mr. Brham Datt for the award of Doctor of Philosophy in Mathematics to the Indian Institute of Technology, New Delhi, is a record of bonafide research work. He has worked for the past three and half years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

N. K. Govil

(N.K. GOVIL)
Assistant Professor
Department of Mathematics
Indian Institute of Technology
New Delhi-110029.

ACKNOWLEDGEMENTS

It is a moment of great pleasure to express my heartiest thanks to my research supervisor Dr. N.K. Govil. His pleasing nature and affectionate encouragement helped me a lot and made my stay here more meaningful. The discussions with him have created an everlasting interest in the subject and helped me in obtaining most of the results reported here. I am extremely grateful to him for his valuable guidance throughout the preparation of this thesis.

I am extremely grateful to Prof. Q.I. Rahman of the University of Montreal, Canada, for his valuable suggestions and interesting discussions during his visit to IIT, Delhi in 1972.

My sincere thanks go to the authorities of the Indian Institute of Technology, Delhi, for providing me with a scholarship and to Prof. K.R. Parthasarathy, Prof. M.P.Singh and Prof. M.K. Jain for their encouragement and for extending all the facilities in the department.

Also my thanks to all my friends and in particular to Dr. V.K. Jain, Dr. Ashok Kumar and Dr. J.S. Rawat who have in one way or the other helped me during my research tenure here.

I would like to take this opportunity to record my sincere gratitude to my wife and parents for their continuous encouragement.

To Miss Neelam Dhody go my thanks for her very elegant preparation of the final typescript.

Brham Datt.

(BRHAM DATT)
Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-110029.

SYNOPSIS

The present thesis entitled "On Extremal Properties and Location of Zeros of Polynomials" consists of two chapters. The first chapter deals with the extremal properties of polynomials and in the second chapter we have studied the problems concerning the location of zeros of polynomials.

If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n with complex coefficients, then according to Bernstein's theorem

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| \qquad (1)$$

and

$$\max_{|z|=R>1} |p(z)| \leq R^n \max_{|z|=1} |p(z)| \qquad (2)$$

Lax [34] and Ankeny and Rivlin [3] have obtained the inequalities analogous to (1) and (2) for polynomials having no zero inside the unit circle. The inequality analogous to (1) for polynomials having no zero inside the circle $|z| = K$, $K \geq 1$ has been obtained by Malik [35]. (Also see Govil and Rahman [21]). Jain [27] considered the class of polynomials having all the zeros either in the left half-plane

or in the right half-plane and satisfying $p(z) \equiv z^n p(\frac{1}{z})$ while O'hara [43] considered the class of polynomials satisfying $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$. For polynomials having all the zeros in $|z| \leq K$, $K \geq 1$, Govil [22] obtained that $\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |p(z)|$. Giroux and Rahman [20] considered the class of polynomials having a prescribed zero on $|z| = 1$ and obtained inequalities analogous to (1) and (2). In Chapter I we consider the class of polynomials satisfying $p(z) \equiv z^n p(\frac{1}{z})$ and replace the hypothesis that polynomial has all its zeros in one half-plane considered by Jain by an alternative hypothesis and obtain a best possible result. Also, we give an example of a polynomial for which the conditions of Jain's theorem are not satisfied while it satisfies the conditions of our theorem. We also obtain inequality analogous to (2) for the class of polynomials having no zero in $|z| < K$, $K \geq 1$. Our result is not best possible but is an improvement over the known results. Besides these, various other results for the polynomials having prescribed zeros have been obtained. Some of these result are best possible.

Let $p(z) = z^n + \sum_{\nu=0}^{n-1} a_\nu z^\nu$ be a polynomial of degree n

A classical result of Cauchy on the location of zeros of

$p(z)$ states that all the zeros lie in $|z| \leq 1+A$ where $A = \max_{0 \leq j \leq n-1} |a_j|$. Joyal, Labelle and Rahman [28] have improved the above mentioned result of Cauchy. They proved that if $\rho = \max_{0 \leq j < n-1} |a_j|$ then all the zeros of $p(z)$ lie in

$$|z| < \frac{1}{2}\left[1 + |a_{n-1}| + \sqrt{(1-|a_{n-1}|)^2 + 4\rho}\right]$$

As a generalization of Eneström-Kakeya Theorem, Joyal, Labelle and Rahman [28] (see also Govil and Rahman [23], Govil and Jain [24]) proved that if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a a polynomial with real coefficients satisfying $a_n \geq a_{n-1} \cdots \geq \ldots \geq a_0$ then all the zeros of $p(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

In the second chapter, we sharpen the above result by obtaining a smaller region and also generalize Cauchy's result. Besides these, various other results concerning the location of zeros of polynomials have also been obtained. Some of our results are best possible.

CONTENTS