SOME FIXED POINT THEOREMS OF NONLINEAR MAPPINGS IN BANACH SPACES

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CONTENTS

CHAPTER Page

CHAPTER Page

IV COMMON FIXED POINTS AND FIXED POINTS OF SET-VALUED MAPPINGS

APPENDIX: SOME MISCELLANEOUS RESULTS

CHAPTER 0

GENERAL INTRODUCTION

If f is a mapping of a topological space X into itself, a point x in X is called a fixed point of f if it satisfies $fx = x$. An important area of mathematical research concerns problems relating to the existence and construction of fixed points of various classes of mappings with domains in various spaces.

The earliest and best-known fixed point theorems include those of Brouwer for continuous mappings in finite dimensional normed spaces, of Schauder which is a generalisation of Brouwer's theorem to infinite dimension and Banach contraction mapping principle for strict contraction in complete metric spaces.

One of the most important classes of mappings is the class of nonexpansive mappings. These are lipschitzian mappings+ with lipschitz constant $k = 1$ in contrast to strict contractions which have $k < 1$. Whereas in the latter case, by the contraction mapping principle, there is a unique fixed point and the successive Picard iterates of any point converges to it, the former is far from so simple - a nonexpansive mapping of a complete metric space into itself may not have a fixed point; if it has, it may not be unique and the Picard sequence may not converge to it.

+ A lipschitzian mapping f is one which satisfies $d(fx,fy) \leqslant k\, d(x,y)$ for any x,y in its domain, where k is some real constant and d is the metric.

The first fixed point theorems of the general type for nonexpansive mappings are those of Browder, Göhde and Kirk who working independently of each other obtained the same result in 1965. However, Kirk's result is slightly stronger in version. While Browder [7] and Göhde's [26] theorem states that any nonexpansive self-mapping of a bounded closed convex subset of a uniformly convex Banach space has a fixed point, Kirk's [42] result says that the same is also true in a reflexive space having normal structure, or equivalently,a nonexpansive self-mapping of a weakly compact convex subset of a Banach space having normal structure has a fixed point.

These results raised considerable interest in nonexpansive mappings and initiated extensive research in fixed point theory of nonlinear mappings in general.

The classical problem still unsolved for nonexpansive mapping is whether or not it has a fixed point if it maps a weakly compact convex subset G of a Banach space into itself. If G has normal structure, then by Kirk's result the answer is 'yes', but normal structure is not an essential condition - this fact having been settled by Karlovitz [38] who has given an example of a reflexive space not having normal structure, yet every nonexpansive self-mapping of a bounded closed convex set in it has a fixed point.

The attempt in Chapter I is towards finding an additional hypothesis on the mapping that should be very mild, yet guarantees existence of fixed point without the normal structure requirement. One such additional hypothesis is requiring the mapping to have "diminishing orbital diameters (d.o.d.)" introduced by Belluce and Kirk in [2] but it is rather too stringent. "Property (A)" which is enjoyed by much wider class of mappings than d.o.d. is introduced. A mapping with d.o.d. admits no periodic points other than the fixed points but a mapping with property (A) can have such priodic points. Some examples to compare d.o.d. with property (A) are presented and theorems with property (A) parallel to those proved by Belluce and Kirk with d.o.d. are proved. A strengthened version of property (A) called property (A') is also introduced to further weaken the geometric assumptions on the domain of the mapping. Property (A') is shown to have similar strong implications as d.o.d. has. The chapter is devoted to the study of the properties (A) and (A') in relation to the existence of fixed points and in comparison to d.o.d. The basic idea of the theorems herecenters around the concepts of asymptotically central set and asymptotic radius of a sequence. These concepts have been extensively studied by several authors, e.g. Garkavi [20], Edelstein [17,19], Rienermann and Schöneberg [65], Anderson, Hyams and McKnight [1], Calder, Coleman and Harris [10], Lim [51] and Reich [63]. An interesting aspect about these concepts is that the asymptotically central set of the orbit of a point under a nonexpansive self-mapping of a weakly compact convex set is itself weakly compact convex and invariant under the mapping.

More general than nonexpansive yet lipschitzian is the class of asymptotically nonexpansive mappings discussed in Chapter II. This class of mappings was introduced by Goebel and Kirk [22] and they have proved that an asymptotically nonexpansive self-mapping of a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Generalisation of this result to all Banach spaces having characteristic of convexity $\epsilon_0 < 1$ follows from another paper of Kirk [44]. However, this generalisation is independently proved here via one Lemma (Lemma 2.2.1). ~~The important and open question whether the result is true for all reflexive spaces having normal structure (like nonexpansive mappings) is also settled affirmatively. However, the proof of this fact appears in Chapter IV, since it follows as a corollary to a theorem in that chapter.~~ The main result in Chapter II is a theorem (Theorem 2.2.1)* on the weak convergence to the fixed point of an asymptotically nonexpansive mapping. This is an extension of the well-known convergence theorem of Opial [57] and

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*This theorem constitutes the paper "Weak convergence to the fixed point of an asymptotically nonexpansive map" to appear in the Proceedings of the American Mathematical Society.

it states that if the space is uniformly convex and has weakly continuous duality mapping, then the iterates of a point under an asymptotically nonexpansive self-mapping of a bounded closed convex subset converges weakly to a fixed point.

In Chapter III several classes of mappings are discussed. Semicontrative mappings which include nonexpansive mapping with completely continuous perturbations was first introduced by Browder in [8]. He has shown (Theorem 3.1.1) that a semicontractive self-mapping of a bounded closed convex subset of a uniformly convex Banach space has a fixed point. This generalises the Browder-Göhde fixed point theorem for nonexpansive mappings. Browder uses the structure of uniformly convex spaces to show that if f is semicontractive and I the identity mapping, then I-f is demiclosed (see Definition 2.1.3). This fact forms the main part in the proof of Theorem 3.1.1. In [47] Kirk has introduced strongly semicontractive mappings, a strengthened version of semicontractiveness and has shown that if f is strongly semicontractive then I-f is demiclosed in a reflexive space. This fact yields a fixed point theorem for strongly semicontractive class of mappings. It is shown in section 3.5 (Theorem 3.5.2) that for a strongly semicontractive f, I-f is demiclosed in any Banach space - reflexivity is not necessary. As such an improved version of Kirk's existence theorem for strongly semicontractive mapping is obtained.

The measures of noncompactness - set-measure of noncompactness introduced by Kuratowski [49] and ball-measure of noncompactness introduced by Sadovsky [66] give rise to set-contractive and ball-contractive mappings which generalise lipschitzian mappings. With Browder's idea of generalising nonexpansive mappings to semicontractive mappings, semi-set-contractive and semi-ball-contractive mappings are introduced which slightly generalise set-contractive and ball-contractive mappings. Some other classes of mappings are also introduced, among them are locally almost 1-set-contractive (la-1-set-contractive) mappings which generalise locally almost nonexpansive (lane) mappings introduced by Nussbaum [55]. All these mappings are studied throughout the chapter. Several new results and generalisations of known results are obtained in the process. For example, it is known and easy to prove that if U is a set-condensing mapping, then I-U is demiclosed. Theorem 3.5.1 shows that it is true even if U is semi-set-condensing. This result in conjunction with Theorem 3.4.1 due to Petryshyn [60] yields Theorem 3.5.4. This is slightly generalised into Theorem 3.5.5 using the notion of demicompact mappings. In Theorem 3.6.1, it is proved that under a semi-set-condensing or a semi-ballcondensing mapping, if the Picard sequence of a point converges weakly to a point ξ, then ξ is a fixed point and the above sequence converges strongly to it. About locally almost 1-set-contractive mappings, the observations made here are similar to those of Nussbaum for lane mappings. It is seen that semi-1-set contractive mappings (introduced earlier) are locally almost 1-set-contractive just as Nussbaum has observed that semicontractive mappings are lane. It is also observed that in a reflexive space locally almost 1-set-contractive mappingsare 1-set-contractive just as lane mappings are. However, the attempt to obtain a fixed point theorem for locally almost 1-set-contractive mappings in a uniformly convex space - an attempt to generalise Nussbaum's fixed point theorem (Theorem 1 of [54] section 4) for lane mappings was unsuccessful. This problem is thrown open.

In Chapter IV, some common fixed point theorems of singlevalued as well as set-valued mappings are proved. Common fixed point theoremsin complete metric spaces for singlevalued mappings have been proved by several authors, e.g. Wong [70], Iseki [33], Husain and Sehgal [32], Srivastava and Gupta [69] and others. Some of these results are recalled in section 4.1. In section 4.2, the object is to take advantage of a lemma due to Goebel, Kirk and Shimi [23] on the geometry of uniformly convex spaces to obtain a common fixed point theorem of two mappings (Theorem 4.2.1) in such spaces. This result is developed into common fixed point theorems for families of mappings (Theorems 4.2.3 and 4.2.4) in the same way as Iseki [33], Husain and Sehgal [32] have done. These theorems* are very general in nature and contain or improve (for domains in uniformly convex spaces) several known results. The corollary to Theorem 4.2.1 contains as a special case the theorem of Goebel, Kirk and Shimi in [24].

For a set-valued mapping f, a point x is called a fixed point of f, if $x \in fx$. In [53] Nadler obtained a fixed point theorem for a set-valued contraction mapping in a complete metric space. Using this technique a common fixed point result is obtained in section 4.3 for two set-valued mappings which are related by some Kannan type inequality.

In section 4.4 a common fixed point theorem for a linearly ordered, commutative semigroup of nonexpansive mappings is proved. Linearly ordered semigroup of mappings and the concept of d.o.d. for such a semigroup were introduced by Mo Tak Kiang in [40]. Property 'A' which was introduced in Chapter I for a nonexpansive mapping is extended in this section to a semigroup of mappings and thereby a common fixed

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*These results comprise the paper "Common fixed points of mappings in a uniformly convex Banach space" published in the Journal of the London Mathematical Society.

point theorem (Theorem 4.4.1) is proved which generalises the results in Chapter I.

In section 4.5 the last section, some common fixed point theorems for eventually nonexpansive semigroup of mappings are obtained. This class of semigroups has recently (1976) been introduced by Kiang [39]. He has proved a common fixed point theorem for such a semigroup in a uniformly convex Banach space (Theorem 4.5.1). This result is generalised in Theorem 4.5.2 to all spaces having characteristic of convexity $\epsilon_0 < 1$. ~~and further generalised in Theorem 4.5.4 to all reflexive spaces having normal structure~~. In Theorem 4.5.3 same result is proved for a compact setting in arbitrary Banach space. In this theorem the semigroup need not be linearly ordered. ~~Corollary 4.5.1 which follows from Theorem 4.5.4 gives affirmative answer to the question raised at the end of section 2.3 on the existence of fixed points of asymptotically nonexpansive mappings~~.

The Appendix contains some miscellaneous results on mappings not considered earlier in the chapters. It contains a fixed point result for mappings which have uniformly continuous Frechet derivative; an elementary proof of a Theorem of Goebel and Kirk [23] on mappings whose iterates have uniform Lipschitz constant(+) and some aspects of Kannan type mappings.

(+) However, the mapping in our proof is assumed to be defined on the whole of the space.