Some Results on the Density of Integral Sets with Missing Differences

by

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To

My Parents

Ram Shiromani and Gayatri

Certificate

This is to certify that the thesis entitled *Some Results on the Density of Integral Sets with Missing Differences* submitted by *Mr. Ram Krishna Pandey* to the Indian Institute of Technology Delhi for the award of the Degree of Doctor of Philosophy is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

A Tripathi

New Delhi

Dr. Amitabha Tripathi
Supervisor

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New Delhi *Ram Krishna Pandey*

Abstract

This thesis deals with an unpublished problem due to T. S. Motzkin concerning the density of integral sets with missing differences. For any set S of nonnegative integers, we denote by $S(n)$ the number of elements $x \in S$ such that $x \leq n$. As usual, we define the upper and lower densities of S (denoted by $\overline{\delta}(S)$ and $\underline{\delta}(S)$, respectively) by

$$\overline{\delta}(S) = \limsup_{n\to\infty} \frac{S(n)}{n}, \qquad \underline{\delta}(S) = \liminf_{n\to\infty} \frac{S(n)}{n}.$$

If $\overline{\delta}(S) = \underline{\delta}(S)$, we denote the common value by $\delta(S)$, and say that S has density $\delta(S)$.

Let M be a given nonempty set of positive integers. A set S of nonnegative integers is called an M-set if a, $b \in S$ implies $a - b \notin M$. In 1970, Motzkin posed the problem of determining the maximal density of M-sets, given by

$$\mu(M) := \sup_S \overline{\delta}(S),$$

where the supremum is taken with respect to the class of all M-sets S. In 1973, Cantor and Gordon settled the problem completely if $|M| \leq 2$. Later on in 1977, Haralambis besides giving few general estimates gave the values of $\mu(M)$ for some sets of the families $\{1, a, b\}$ and $\{1, 2, a, b\}$. Later, Gupta in her Ph.D. thesis improved a result by Haralambis on $\{1, a, b\}$ and gave a lower bound for $\mu(M)$ for $M = \{a, b, a+b\}$. She also gave the expressions for $\mu(M)$ for $M = \{a, b, a+b, 2b\}$

where $b \equiv a + 2 \pmod 3$ and for $M = \{a, b, a+b, 2a\}$ where $b \equiv a + 1 \pmod 3$. Gupta and Tripathi also gave expression for $\mu(M)$ when the elements of M are in arithmetic progression. The problem is not solved, in general, if $|M| \geq 3$. Using some colouring problems in Graph Theory, Liu and Zhu besides giving some bounds for a certain type of Almost Difference Closed Set gave values for all other types of Almost Difference Closed Sets. They also gave the value of $\mu(M)$ when M is the union of two intervals.

In this thesis, expressions and bounds for $\mu(M)$ for some families M with three or more elements has been obtained. The three element families considered are: $\{a, b, n(a+b)\}$, $\{a, b, c\}$ where $c = nb$ or na with $a < b$, and $\{a, b, a+nb\}$ with $a < b$. The family $\{a, b, n(a+b)\}$ generalizes $\{a, b, a+b\}$ already considered by Haralambis, Gupta and Liu & Zhu. The family $\{1, b, c\}$ already discussed by Haralambis can be obtained as a particular case of the family $\{a, b, na\}$ by putting $a = 1$. The four element families discussed are $\{a, b, a+b, nb\}$ and $\{a, b, a+b, na\}$, generalizing the families $\{a, b, a+b, 2b\}$ and $\{a, b, a+b, 2a\}$ in all the cases. The general cases deal with sets M that are related to arithmetic progressions. More specifically, we consider the cases when either some elements have been removed from an arithmetic progression or some elements have been added to an arithmetic progression. The Lonely Runner Conjecture by Wills is also discussed as a part of the general case.

Contents

Certificate i

Acknowledgements iii

Abstract v

1 Introduction 9

1.1 The Problem 10

1.2 Graph Theoretic Equivalence 16

2 Some Families of Cardinality 3 23

2.1 The family $M = \{a, b, n(a + b)\}$ 23

2.2 The family $M = \{a, b, c\}$, where $c = nb$ or na 34

2.3 The family $M = \{a, b, a + nb\}$ for sufficiently large n 64

3 Some Families of Cardinality 4 79

4 Some General Results 95

4.1 Gaps in an Arithmetic Progression 96

4.2 Almost Arithmetic Progression 111

5 Some Remarks Concerning the Lonely Runner Conjecture 117

6 Concluding Remarks **125**

Bibliography **129**

Bio-Data **133**