DIFFERENCE SCHEMES FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

By

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CERTIFICATE

This is to certify that the thesis entitled, "Difference Schemes for Parabolic Partial Differential Equations" which is being submitted by Mr. Abdul Gani Lone for the award of the degree, Doctor of Philosophy (Mathematics), to the Indian Institute of Technology, Delhi, is a record of bonafide research work. He has worked for the last three years and four months under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

M.K. Jain 29/11/76
(Professor M.K. Jain)

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SYNOPSIS

In recent years the difference schemes for the solution of initial and initial boundary value problems for partial differential equations of parabolic type have developed considerably. Many of the partial differential equations encountered by scientists and engineers can be solved in a satisfactory manner by the finite difference methods. The aim of this thesis is to construct higher order, stable difference schemes which produce more accurate results than the present methods.

The thesis is divided into the following four chapters:

Chapter I: Explicit-Implicit Difference Schemes of the Heat Conduction Equation in One Space Dimension.

Chapter II: Multilevel ADI Difference Schemes for the Heat Conduction Equation and Dirichlet Problem in Two and Three Space Dimensions.

Chapter III: Multilevel ADI Methods for Parabolic Partial Differential Equation With Variable Coefficients.

Chapter IV: High Order Difference Formulae for a Fourth Order Parabolic Partial Differential Equation.

In Chapter I some new three level implicit and explicit schemes are derived for the one dimensional heat conduction equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. The general three level explicit (E) and Implicit (I) schemes have the form

$$E: \quad \frac{\nabla_t + \tau_1^* \nabla_t^2}{1 - \gamma_1^* \nabla_t} u_i^{n+1} = r\delta_x^2 u_i^n$$

$$I: \quad \frac{\nabla_t + \tau_1 \nabla_t^2}{1-\gamma_1\nabla_t + \gamma_2\nabla_t^2} u_i^{n+1} = r\delta_x^2 u_i^{n+1}$$

respectively. Where τ_1^*, γ_1^*, $\tau_1, \gamma_1, \gamma_2$ are parameters which can be chosen according to stability and accuracy requirements. Stability diagrams have been drawn in both the cases. These two general three level schemes when applied in an Explicit-Implicit manner are shown to be equivalent to some five level explicit scheme. With a proper choice of parameters, some interesting schemes have resulted. If $\tau_1 = \frac{1}{2} - \gamma_1$, $\gamma_2 = \frac{1}{2}\gamma_1$; $\tau_1^* = -\frac{1}{2}$, $\gamma_1^* = 0$ and γ_1 arbitrary, the Explicit-Implicit scheme reduces to a five level explicit scheme, which is stable for $r \leqslant \frac{1}{2(1-\gamma_1)}$. For optimal value of γ_1 and for larger values of r the E-I method gives most accurate results among the explicit methods.

In Chapter II we have given a general method to obtain multilevel difference schemes for the heat conduction equation $\frac{\partial u}{\partial t} = \sum_{i=1}^{p} \frac{\partial^2 u}{\partial x_i^2}$ in the P space variables $x_1, x_2, \ldots, x_p$ with appropriate initial and boundary conditions. A consistant multilevel implicit difference scheme is given by

$$\left[\frac{\nabla_t + \sum_{m=1}^{q} \tau_m \nabla_t^{m+1}}{1 + \sum_{j=1}^{s} \gamma_j \nabla_t^{j}} \right] u^{n+1} = r \left[\sum_{\alpha=1}^{p} \frac{\delta_{x_\alpha}^2}{1 + a_1 \delta_{x_\alpha}^2} \right] u^{n+1}$$

where $q \gtrless s$ and τ_m's, γ_j's and a_1 are arbitrary parameters. This difference scheme represents a $(q+2)$-level scheme if $q \geqslant s$ and $(s+1)$ level scheme if $q < s$. With proper choice of τ_m's, γ_j's, q,s and a_1 stable multilevel ADI difference schemes with order of accuracy $O(k^{q+s+1} + h^2)$ and $O(k^{q+s+1} + h^4)$ have been obtained. The application of some three level difference schemes of the parabolic equation, for the solution of the steady state elliptic problem in two and three space dimensions is discussed. Some optimal three level schemes are suggested from the computational results. We also propose a new set of iteration parameters viz.,

$$r_\mu = \frac{1}{16\gamma_2\, \xi_r^{(\mu)}} \quad , \quad \xi_r^{(1)} = \frac{(1 + Q\delta)^2}{1 + \delta} \sin^2 \frac{\pi h}{2}$$

$$\xi_r^{(\mu)} = (1 + Q\delta)^{2(\mu-1)}\, \xi_r^{(1)}$$ where γ_2 is an arbitrary parameter

$$\delta = 0.1 \quad \text{and} \quad 2 \leqslant Q \leqslant 6.$$

This new set of parameters considerably reduces the theoretical bound on the number of iteration cycles required to attain an accuracy of 10^{-6}. Two numerical examples are computed with

Hadjidimos and the new parameters to show the superiority of our schemes.

In Chapter III we have derived multilevel alternating direction implicit (ADI) methods for the solution of the parabolic partial differential equation of the type

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{q} a_i(x_1,x_2,\ldots,x_i;\ t)\ \frac{\partial^2 u}{\partial x_i^2}$$

$$a_i(x_1,\ x_2,\ldots,\ x_i,t) > 0 \qquad i = 1,2,\ldots,q$$

with boundary conditions

$$u(x_1,x_2,\ldots,x_q,0) = f_1(x_1,x_2,\ldots,x_q),(x_1,x_2,\ldots,x_q) \in R$$

$$u(x_1,x_2,\ldots,x_q,t) = f_2(x_1,x_2,\ldots,x_q;t) \text{ on } \Gamma, 0 \leqslant t \leqslant T$$

where R is a rectangular connected region and Γ its boundary. For example, in one space dimension we have the heat equation with variable coefficient as

$$\frac{\partial u}{\partial t} = a_1(x_1,t)\ \frac{\partial^2 u}{\partial x_1^2}$$

Applying it on the step (n+1, j) we can write

$$-\log(1-\nabla_t)u_j^{n+1} = 4r(a_1(x_1,t))_j^{n+1}(\sinh^{-1}\frac{\delta_x}{2})^2\ u_j^{n+1}$$

Approximating $\log(1-\nabla_t) \simeq \nabla_t,\ \nabla_t + \frac{1}{2}\nabla_t^2$ and

$4(\sinh^{-1} \frac{\delta_x}{2})^2 \simeq \frac{\delta^2_{x_1}}{1 + \frac{1}{12}\delta^2_{x_1}}$ we have unconditionally stable two and three level schemes

$$(1 + X_1 - X_2)u^{n+1} = (1 + X_1)u^n$$

$$(1 + X_1 - \tfrac{2}{3} X_2)u^{n+1} = \tfrac{1}{3}(1 + X_1)(4u^n - u^{n-1})$$

of order $O(k + h^4)$ and $O(k^2 + h^4)$ respectively, where $X_1 = \frac{1}{12} a_1 \delta^2_{x_1} a_1^{-1}$, $X_2 = r a_1 \delta^2_{x_1}$. In two and three space dimensions these schemes take the form

$$(1 + X_1 - X_2)(1 + Y_1 - Y_2)u^{n+1} = (L_1 + R_1)u^n$$

$$(1 + X_1 - X_2)(1 + Y_1 - Y_2)(1 + Z_1 - Z_2)u^{n+1} = (L_2 + R_2)u^n$$

and

$$(1 + X_1 - \tfrac{2}{3} X_2)(1 + Y_1 - \tfrac{2}{3} Y_2)u^{n+1} = \tfrac{1}{3}(L_1 + \tfrac{4}{9} X_2 Y_2)(4u^n - u^{n-1}) + \tfrac{2}{3} R_1 (2u^n - u^{n-1})$$

$$(1 + X_1 - \tfrac{2}{3} X_2)(1 + Y_1 - \tfrac{2}{3} Y_2)(1 + Z_1 - \tfrac{2}{3} Z_2)u^{n+1} = \tfrac{1}{3}(L_2 - \tfrac{4}{9} R_3)(4u^n - u^{n-1}) + \tfrac{2}{3} R_2 (2u^n - u^{n-1})$$

where

$$L_1 = (1 + X_1)(1 + Y_1), \quad R_1 = Y_1 X_2 - X_2 Y_1,$$

$$L_2 = (1 + X_1)(1 + Y_1)(1 + Z_1)$$

$$R_2 = Y_1 X_2 - X_2 Y_1 + Z_1 X_2 - X_2 Z_1 + Z_1 Y_2 - Y_2 Z_1$$

$$R_3 = (1 + X_1) Y_2 Z_2 + X_2 (1 + Y_1) Z_2 + X_2 Y_2 (1 + Z_1)$$

and $$Y_1 = \frac{1}{12} a_2 \delta^2_{x_2} a_2^{-1} , \quad Y_2 = r a_2 \delta^2_{x_2}$$

$$Z_1 = \frac{1}{12} a_3 \delta^2_{x_3} a_3^{-1} , \quad Z_2 = r a_3 \delta^2_{x_3} .$$

These schemes are not only unconditionally stable, but also retain their order of accuracy i.e. $O(k + h^4)$ and $O(k^2 + h^4)$ respectively. Similarly by taking higher order polynomial approximation to $\log(1-\nabla_t)$ and same approximation to $(\sinh^{-1} \frac{\delta_x}{2})^2$ unconditionally stable formulas are derived upto order $O(k^6)$. These formulas have then been extended to two and three space variables and their split forms are obtained. Widlund's Analysis is used to prove the stability of these schemes. Computations are performed in two examples and the advantages of using the new schemes are given.

In Chapter IV the fourth order parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

with proper initial and boundary conditions, which occurs in the study of the transverse vibrations of a uniform flexible beam is studied. If the bending moment is not prescribed at the two end points $x = 0$ and L then we need to derive a direct difference scheme for its solution. The direct difference scheme to the above differential equation may be written as

$$[1 + \beta\delta_x^2 + (\beta_1 + \tau r^2)\delta_x^4]\delta_t^2 u_j^n + r^2\delta_x^4 u_j^n = 0$$

where β_1, β and τ are parameters to be chosen according to stability and accuracy requirements. Some higher accuracy schemes of order $O(h^2 + k^4)$, $O(h^4 + k^4)$ are obtained. Todd's and Crandall's schemes are deduced as particular cases. A numerical example is computed and the results indicate that these difference schemes behave very efficiently and are more accurate than Crandall's and Todd's schemes.

If the bending moment is prescribed at the two end points $x = 0$ and L, then the Richtmyer's approach can be followed. We reduce the fourth order partial differential equation to a system of first order equations of the form

$$\frac{\partial v}{\partial t} = -\frac{\partial^2 w}{\partial x^2} \quad , \quad \frac{\partial w}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

where $\frac{\partial u}{\partial t} = v$ and $\frac{\partial^2 u}{\partial x^2} = w$ and then construct numerical methods for solving such a system. Keeping γ_1 and γ_2 as arbitrary parameters, the following set of difference equation may be obtained.

$$\begin{aligned}(3-2\gamma_1)(1+\tfrac{1}{12}\delta_x^2)v_j^{n+1} = {} & 4(1-\gamma_1)(1+\tfrac{1}{12}\delta_x^2)v_j^n - (1-2\gamma_1)(1+\tfrac{1}{12}\delta_x^2)v_j^{n-1} \\ & -2r(1-\gamma_1+\gamma_2)\delta_x^2 w_j^{n+1} - 2r(\gamma_1-2\gamma_2)\delta_x^2 \\ & w_j^n - 2\gamma_2 r\delta_x^2 w_j^{n-1}\end{aligned}$$

$$\begin{aligned}(3-2\gamma_1)(1+\tfrac{1}{12}\delta_x^2)w_j^{n+1} = {} & 4(1-\gamma_1)(1+\tfrac{1}{12}\delta_x^2)w_j^n - (1-2\gamma_1)(1+\tfrac{1}{12}\delta_x^2)w_j^{n-1} \\ & +2r(1-\gamma_1+\gamma_2)\delta_x^2 v_j^{n+1} + 2r(\gamma_1-2\gamma_2)\delta_x^2 \\ & v_j^n + 2r\gamma_2\,\delta_x^2 v_j^{n-1}\end{aligned}$$

This three level scheme is stable if $\gamma_1 \leqslant 1$ and $1-2\gamma_1 + 4\gamma_2 \geqslant 0$ and is of order $O(k^2 + h^4)$. For different values of γ_1 and γ_2 these schemes are applied to a numerical example and it is shown that there are many values of (γ_1, γ_2) on $1-2\gamma_1 + 4\gamma_2 = 0$ which give better results than two level schemes. In both cases stability diagrams have been drawn. Also difference schemes for the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \nabla^4 u = 0$$

subject to appropriate initial and boundary conditions are derived.

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