A THESIS
ON
VARIABLE MESH DIFFERENCE METHODS FOR THE SOLUTION OF TWO POINT SINGULAR PERTURBATION BOUNDARY VALUE PROBLEMS

By

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CERTIFICATE

This is to certify that the thesis entitled, 'Variable Mesh Difference Methods for the Solution of Two-Point Singular Perturbation Boundary Value Problems', which is being submitted by Mr. G.S. Subramanyam for the award of the degree, Doctor of Philosophy (Mathematics), to the Indian Institute of Technology, Delhi, is a bonafide record of research work done under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other university or Institute for the award of any degree or diploma.

(S.R.K. Iyengar)

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SYNOPSIS

Two point boundary value problems of the form

$$f(x,y,y',\varepsilon y'') = 0 \tag{1a}$$

$$g(x,y,y', y'', \varepsilon y''') = 0 \tag{1b}$$

$$F(x,y,y',y'', y''', \varepsilon y^{iv}) = 0 \tag{1c}$$

in which the highest order derivative is multiplied by a small positive parameter ε, arise in fluid mechanics (e.g. boundary layer problems), elasticity (eg. edge effect in shells, load deflection problems), quantum mechanics (WKB problems) etc. One is often interested in the asymptotic behaviour of the solution function $y(x)$ as $\varepsilon \to 0$. Such problems are often referred to as singular perturbation problems. Often, these differential equations are difficult to solve numerically because of accuracy and stability problems associated with the eigen values of widely different magnitudes. Difference methods with constant mesh length usually fall in these problems. For example, consider the problem $\varepsilon y'' + y' = 0$, $x \in [-1,1]$, $y(-1) = 1$, $y(1) = 2$. There is a boundary layer of order $O(\varepsilon)$ near $x = -1$. With $\varepsilon = 10^{-8}$ one would need a mesh spacing of order 10^{-10} near $x = -1$ while 10^{-2} would be adequate elsewhere. In this thesis, an attempt is made to construct variable mesh difference methods for the numerical solution of(1). Such methods can avoid the difficulties that arise in the use of constant mesh difference methods. The thesis is divided into four chapters.

CHAPTER 1: REVIEW OF COMPUTATIONAL METHODS FOR THE SOLUTION OF TWO POINT BOUNDARY VALUE PROBLEMS

A review of literature concerning the solution of the two point singular perturbation boundary value problems (1) is given. The utility of the variable mesh methods for the solution of (1) is described.

CHAPTER 2: COMPUTATIONAL METHODS FOR SECOND ORDER SINGULAR PERTURBATION BOUNDARY VALUE PROBLEMS

In this chapter, variable mesh difference methods of third and fifth orders are constructed for the solution of the second order differential equation

$$y'' = f(x,y,y'),\ x\epsilon\ [a,b] \tag{2}$$

Boundary conditions of the following forms are considered.

$$\text{i)}\ y(a) = \alpha, \quad y(b) = \beta \tag{2a}$$

$$\text{ii)}\ K_1 y(a) - K_2 y'(a) = A$$

$$K_3 y(b) + K_4 y'(b) = B \tag{2b}$$

$$\text{iii)}\ h_1(y(a), - y'(a)) = 0,\ h_2(y(b),y'(b)) = 0 \tag{2c}$$

The interval [a,b] is divided into a number of subintervals using the variable spacing $h_j = x_j - x_{j-1}$ and $\sigma_j = h_{j+1}/h_j$. A one-parameter family of third order methods for the solution of (2) is constructed. Two methods of third order which are optimal in the sense that they have smaller truncation error are deduced When $\sigma_j = 1$, the method reduces to the constant mesh case [see Allen (1966), Steplemann (1976) and Chawla (1978)]. The method is then of order four. The convergence of the above third order method is shown. A one parameter family of fifth order methods is also constructed for the solution of (2). Again, an optimal method in the sense that it has smaller truncation error is deduced. When $\sigma_j = 1$, the constant mesh case, this method issimplified and reduces to a method of order six [Chawla (1979)]. Suitable difference approximations to the boundary conditions are derived. The coefficient matrix of the system produced by both the third order and fifth order methods is tridiagonal. If the differential equation is linear, then the system of equations is linear, otherwise it will be nonlinear. These non-linear equations can be solved by the Newton-Raphson scheme.

CHAPTER-3: COMPUTATIONAL METHODS FOR THE SOLUTION OF HIGHER ORDER DIFFERENTIAL EQUATIONS

In this chapter, variable mesh difference approximations are constructed for the following boundary value problems

i) $$y''' = f(x,y,y',y'') \tag{3.1a}$$

$$y(a) = \alpha,\ y'(a) = \beta,\ y(b) = \gamma \tag{3.1b}$$

ii) $$y^{iv} = f(x,y,y',y'') \tag{3.2a}$$

$$y(a) = \alpha,\ y'(a) = \beta,\ y(b) = \gamma,\ y'(b) = \delta \tag{3.2b}$$

iii) $$y^{iv} = f(x,y,y',y'',y''') \tag{3.3a}$$

$$y(a) = \alpha,\ y'(a) = \beta,\ y(b) = \gamma,\ y'(b) = \delta \tag{3.3b}$$

The same discretization as in Chapter 2 is used. A variable mesh, four point, and a unique third order scheme for the solution of (3.1) is constructed. A five point, one parameter family of fourth order schemes is also derived. This parameter is again determined such that the truncation error is small. The expressions for the truncation error are given. The four point scheme gives a four diagonal matrix system for the solution of (3.1) while a five point scheme gives a five diagonal scheme. We suggest for computational purpose the four point scheme as it uses minimum number of nodal points required for the solution of (3.1). Suitable difference approximations at the boundary are derived.

A one parameter family of five point third order schemes for the solution (3.2) is derived. By putting the leading term of the truncation error to zero, we produce a unique fourth order scheme for the solution of (3.2). The truncation error of the formula is given. This scheme gives rise to a five diagonal system. This method is applied on the scalar test equation $y^{iv} = Ky''$ and the stability of the method is studied.

A one parameter family of five point third order methods is derived for the solution of (3.3). This parameter may be determined such that the formula is optimal in the sense that the truncation error is smaller.

CHAPTER 4: COMPUTATIONAL EXPERIMENTS

In this chapter, first the implementation of the variable mesh methods is discussed. The methods derived in Chapters 2 and 3 are applied to the following problems.

EXAMPLE 1 Convection-diffusion problem

$$y'' = \beta y', \quad y(0) = 1, \; y(1) = 0$$

Let N* be the total number of mesh points.
This problem is solved for N* = 8, 11, 16 etc. and for various values of β = 10, 50, 100, 150. For large β, optimal third order method produces better results than the constant mesh case. The sixth order constant mesh method exhibited oscillations for β = 100, 150, while the optimal fifth order variable mesh method produced oscillation free and accurate solutions.

EXAMPLE 2 $\varepsilon y'' + 2y' + y = -3, \quad y(-1) = 1, \; y(1) = 2$

There is a boundary layer near $x = -1$. Computations are performed using $\varepsilon = 10^{-6}$. In the constant mesh case the error is too large near $x = -1$, even with N* = 101 or N* = 601. Using the optimal third order method I, with N* = 101 we obtained 10^{-3} accuracy and with N* = 601, we achieved 10^{-4} accuracy.

EXAMPLE 3 $\varepsilon y'' + xy' - .01 y = 0, \; y(-1) = 1, \; y(1) = 2$

In this problem, the boundary layer exists near the origin $x = 0$. We choose $\varepsilon = 10^{-6}$. The fourth order constant mesh scheme produced severe oscillations at almost every node for $N^* = 101$ and $N^* = 201$. On the other hand, the optimal third order variable mesh method I produced oscillation free solutions and the results are accurate up to 3 significant digits. The fifth order method produced inferior results compared to the third order method.

EXAMPLE 4 Load-deflection problem

$$\varepsilon y^{iv} - y'' = p(x)$$

$$y(0) = 0 = y'(0) , \; y(1) = 0 = y'(1)$$

we choose $p(x) = x, x^2$ and $\varepsilon = 10^{-4}$. Even though the boundary layer may exist at both ends, both equal and variable mesh methods produced accurate results. Computations are performed with $N^* = 101$ and $N^* = 201$

EXAMPLE 5 Burger's equation

$$u'' = \frac{1}{\nu}(u-\beta)u', \; u(-\infty) = 1.0, \; u(\infty) = 0.0$$
$$\beta = 0.5,$$

The computations are performed with/$N^* = 15$ and $N^* = 31$. The resulting nonlinear equations are solved by the Newton-Raphson scheme. We find that the optimal third order variable mesh solutions are superior to the fourth order constant mesh solutions. For small values of ν, it is found that the sixth order constant mesh method produced solutions which exhibit oscillations, whereas the fifth order variable mesh method produced accurate and oscillation free solutions.

EXAMPLE 6 $\varepsilon y'' + y'^2 = 1$, $y(0) = y(1) = 1$

We choose $\varepsilon = 10^{-6}$. For $N^* = 401$, the fourth order constant mesh method produced solutions with 10^{-1} accuracy only whereas the third order variable mesh method I produced solutions which have 10^{-6} accuracy.

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C O N T E N T S