A STUDY ON HOMOLOGY MANIFOLDS AND NORMAL PSEUDOMANIFOLDS

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A Study on Homology Manifolds and Normal Pseudomanifolds

by

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Dedicated to

My Parents:

Shri Kamal Kumar Sarkar
and
Smt Bela Dey Sarkar

Certificate

This is to certify that the thesis entitled **"A Study on Homology Manifolds and Normal Pseudomanifolds"** submitted by **Mr. Sourav Sarkar** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

The combinatorial characterization of certain special classes of simplicial complexes, specifically homology manifolds and normal pseudomanifolds, is being developed in terms of the g-theorem. The variation of the third component of the g-vector of a d-dimensional simplicial complex Δ, denoted as $g_2(\Delta) := f_1(\Delta) - (d+1)f_0(\Delta) + \binom{d+2}{2}$, where $f_0(\Delta)$ and $f_1(\Delta)$ represent the number of vertices and edges in Δ, respectively, has played a significant role in this development.

A *homology d-manifold* is a pure simplicial complex in which the link of every vertex is a homology $(d-1)$-sphere. A *normal d-pseudomanifold* is a strongly connected pure simplicial complex in which every face of dimension $d-1$ is contained in exactly two facets and the links of all the simplices of dimensions up to $d-2$ are connected. In a normal d-pseudomanifold Δ, the link of a vertex is a normal $d-1$-pseudomanifold. Vertices in Δ whose links are triangulated spheres are called *non-singular* vertices, while the remaining are called *singular* vertices.

Given a normal d-pseudomanifold K, the well-known lower bound theorem asserts that $g_2(K) \geq g_2(\mathrm{lk}_K \sigma)$ for every simplex σ of codimension three or more. A normal d-pseudomanifold K is called g_2-minimal with respect to a vertex v if $g_2(K) = g_2(\mathrm{lk}_K v)$ holds. Let Δ be a g_2-minimal normal 3-pseudomanifold. When Δ contains at most two singular vertices, its combinatorial characterization is known [17]. In Chapter 3, we present a combinatorial characterization of such a Δ when it has three singular vertices, including one $\mathbb{RP}^2$-singularity, or four singular vertices, including two $\mathbb{RP}^2$-singularities. In both cases, we prove that Δ is obtained from a one-vertex suspension of a surface, and some boundary complexes of 4-simplices by applying the combinatorial operations of types connected sums, vertex foldings, and edge folding.

It is known that when $g_2 = 0$, all normal pseudomanifolds of dimensions at least three are stacked spheres. In the cases of $g_2 = 1$ and 2, all homology manifolds are polytopal spheres and can be obtained through retriangulation or join operations from the previous ones. In Chapter 4, we provide a combinatorial characterization of the homology d-manifolds, where $d \geq 3$ and $g_2 = 3$. These are spheres and can be obtained through operations such as joins, some retriangulations, and connected sums from spheres with $g_2 \leq 2$. Furthermore, we have presented a structural result on prime normal d-pseudomanifolds with $g_2 = 3$.

In Chapter 5, the homology 4-manifolds with $g_2 \leq 5$ are characterized combinatorially. It is well-known that all homology 4-manifolds for $g_2 \leq 2$ are polytopal spheres. We demonstrate that homology 4-manifolds with $g_2 \leq 5$ are triangulated spheres and are derived from triangulated 4-spheres with $g_2 \leq 2$ by a series of connected sum, bistellar 1- and 2-moves, edge contraction, edge expansion, and edge flipping operations. We establish that the above inequality is optimally attainable, i.e., it cannot be extended to $g_2 = 6$.

Finally, we worked with balanced normal pseudomanifolds in Chapter 6. For a balanced normal d-pseudomanifold, a PL invariant is introduced, known as the balanced genus. We establish a fundamental result: a 3-manifold M is homeomorphic to a sphere if and only if its balanced genus $\mathcal{G}_M$ is bounded by 3. In cases where M is a 3-manifold distinct from a sphere, then $\mathcal{G}_M$ assumes a lower bound of $m+3$, where m denotes the rank of the fundamental group of M. Transitioning to 4-manifolds, we prove that a 4-manifold M is PL-homeomorphism to a sphere if and only if its balanced genus $\mathcal{G}_M$ is bounded by $2\chi(M) + 10$. If M is a 4-manifold that is not a sphere, we prove that $\mathcal{G}_M$ is bounded from below by $2\chi(M) + 5m + 11$, where m denotes the rank of the fundamental group of M.

सार

सिंप्लिशियल कॉम्प्लेक्सों की कुछ विशेष श्रेणियों, विशेष रूप से होमोलॉजी मैनिफोल्ड्स और नॉर्मल स्यूडोमैनिफोल्ड्स की संयोजनात्मक विशेषताओं को g-प्रमेय के संदर्भ में विकसित किया जा रहा है। किसी d-आयामी सिंप्लिशियल कॉम्प्लेक्स Δ के g-वेक्टर के तीसरे घटक, जिसे $g_2(\Delta) := f_1(\Delta) - (d+1)f_0(\Delta) + \binom{d+2}{2}$ से निरूपित किया जाता है, जहाँ $f_0(\Delta)$ और $f_1(\Delta)$ क्रमशः Δ में शीर्षों और किनारों की संख्या दर्शाते हैं, ने इस विकास में महत्वपूर्ण भूमिका निभाई है।

होमोलॉजी d-मैनिफोल्ड एक शुद्ध सिंप्लिशियल कॉम्प्लेक्स होता है जिसमें प्रत्येक शीर्ष का लिंक एक होमोलॉजी $(d-1)$-स्फ़ीयर होता है। नॉर्मल d-स्यूडोमैनिफोल्ड एक मजबूती से जुड़ा हुआ शुद्ध सिंप्लिशियल कॉम्प्लेक्स होता है जिसमें हर $(d-1)$-आयामी फेस ठीक दो फेसेट्स में सम्मिलित होता है, और आयाम $(d-2)$ तक के सभी सिंप्लेक्सों के लिंक जुड़े हुए होते हैं। किसी नॉर्मल d-स्यूडोमैनिफोल्ड Δ में किसी शीर्ष का लिंक एक नॉर्मल $(d-1)$-स्यूडोमैनिफोल्ड होता है। Δ में वे शीर्ष जिनके लिंक ट्रायंगुलेटेड स्फ़ीयर होते हैं, उन्हें नॉन-सिंगुलर शीर्ष कहा जाता है, जबकि शेष को सिंगुलर शीर्ष कहा जाता है।

यदि K एक नॉर्मल d-स्यूडोमैनिफोल्ड है, तो प्रसिद्ध लोअर बाउंड प्रमेय कहता है कि $g_2(K) \geq g_2(\mathrm{lk}_K \sigma)$, हर उस सिंप्लेक्स σ के लिए जिसके कोडायमेंशन तीन या अधिक हो। एक नॉर्मल d-स्यूडोमैनिफोल्ड K को किसी शीर्ष v के सापेक्ष g_2-मिनिमल कहा जाता है यदि $g_2(K) = g_2(\mathrm{lk}_K v)$ होता है। मान लीजिए कि Δ एक g_2-मिनिमल नॉर्मल 3-स्यूडोमैनिफोल्ड है। जब Δ में अधिकतम दो सिंगुलर शीर्ष होते हैं, तब इसका संयोजनात्मक वर्णन [17] में दिया गया है। अध्याय 3 में, हम ऐसे Δ का संयोजनात्मक वर्णन प्रस्तुत करते हैं जिसमें या तो तीन सिंगुलर शीर्ष (जिनमें से एक $\mathbb{RP}^2$ सिंगुलैरिटी हो), या चार सिंगुलर शीर्ष (जिनमें से दो $\mathbb{RP}^2$ सिंगुलैरिटीज़ हों) होते हैं। दोनों ही मामलों में, हम सिद्ध करते हैं कि Δ को किसी सतह की एक-शीर्ष सस्पेंशन और कुछ 4-सिंप्लेक्सों की बाउंड्री कॉम्प्लेक्सों से संयोजनात्मक क्रियाओं जैसे कि कनेक्टेड समों, वर्टेक्स फ़ोल्डिंग्स, और एज फ़ोल्डिंग्स के द्वारा प्राप्त किया गया है।

यह ज्ञात है कि जब $g_2 = 0$ होता है, तो सभी नॉर्मल स्यूडोमैनिफोल्ड्स (आयाम ≥ 3) स्टैक्ड स्फ़ीयर्स होते हैं। $g_2 = 1$ और $g_2 = 2$ के मामलों में, सभी होमोलॉजी मैनिफोल्ड्स पॉलिटोपल स्फ़ीयर्स होते हैं और उन्हें पिछले कॉम्प्लेक्सों से रीट्रायंगुलेशन या जॉइन ऑपरेशन के माध्यम से प्राप्त किया जा सकता है। अध्याय 4 में, हम $d \geq 3$ और $g_2 = 3$ वाले होमोलॉजी d-मैनिफोल्ड्स का संयोजनात्मक वर्णन प्रदान करते हैं। ये सभी स्फ़ीयर्स होते हैं और उन्हें $g_2 \leq 2$ वाले स्फ़ीयर्स से जॉइन ऑपरेशन्स, कुछ रिट्रायंगल्यूलेशन्स, और कनेक्टेड समों जैसे ऑपरेशन्स के माध्यम से प्राप्त किया जा सकता है। इसके अतिरिक्त, हमने $g_2 = 3$ वाले प्राइम नॉर्मल d-स्यूडोमैनिफोल्ड्स पर एक संरचनात्मक परिणाम भी प्रस्तुत किया है।

अध्याय 5 में, $g_2 \leq 5$ वाले होमोलॉजी 4-मैनिफोल्ड्स का संयोजनात्मक वर्णन किया गया है। यह सुविख्यात है कि $g_2 \leq 2$ होने पर सभी होमोलॉजी 4-मैनिफोल्ड्स पॉलिटोपल स्फ़ीयर्स होते हैं। हम प्रदर्शित करते हैं कि $g_2 \leq 5$ वाले होमोलॉजी 4-मैनिफोल्ड्स ट्रायंगुलेटेड स्फ़ीयर्स होते हैं और उन्हें $g_2 \leq 2$ वाले ट्रायंगुलेटेड 4- स्फ़ीयर्स से कनेक्टेड समों, बाइस्टेलर 1- और 2-मूव्स, एज कॉन्ट्रैक्शन्स, एज एक्सपैंशन्स, और एज फ़्लिपिंग्स जैसी ऑपरेशन्स की एक श्रृंखला द्वारा प्राप्त किया जा सकता है। हम यह भी स्थापित करते हैं कि यह इनेक्वैलिटी सर्वोत्तम है, अर्थात इसे $g_2 = 6$ तक विस्तारित नहीं किया जा सकता।

अंततः, अध्याय 6 में हमने बैलेंस्ड नॉर्मल स्यूडोमैनिफोल्ड्स के साथ कार्य किया। किसी बैलेंस्ड नॉर्मल d-स्यूडोमैनिफोल्ड के लिए एक PL इनवेरिएंट परिभाषित किया गया है, जिसे "बैलेंस्ड जीनस" कहा जाता है। हम एक मौलिक परिणाम

स्थापित करते हैं: कोई 3-मैनिफोल्ड M एक स्फ़ीयर के होमियोमॉर्फिक होता है यदि और केवल यदि उसका बैलेंस्ड जीनस $\mathcal{G}_M \leq 3$ हो। उन परिस्थितियों में जब M एक स्फ़ीयर से भिन्न 3-मैनिफोल्ड होता है, तब $\mathcal{G}_M$ का न्यूनतम मान $m+3$ होता है, जहाँ m, M के फंडामेंटल ग्रुप की रैंक को दर्शाता है। 4-मैनिफोल्ड्स के संदर्भ में, हम सिद्ध करते हैं कि कोई 4-मैनिफोल्ड M एक स्फ़ीयर से PL-होमियोमॉर्फिक होता है यदि और केवल यदि उसका बैलेंस्ड जीनस $\mathcal{G}_M \leq 2\chi(M) + 10$ हो, जहाँ $\chi(M)$ मैनिफोल्ड का ऑयलर कैरैक्टरिस्टिक है। यदि M एक ऐसा 4-मैनिफोल्ड है जो स्फ़ीयर नहीं है, तो हम सिद्ध करते हैं कि $\mathcal{G}_M$ का न्यूनतम मान $2\chi(M) + 5m + 11$ होता है, जहाँ m फिर से M के फंडामेंटल ग्रुप की रैंक को दर्शाता है।

Contents

List of Symbols

Symbol	**Meaning**
$\mathbb{R}^n$	The n-dimensional Euclidean Space,
$\mathbb{N}$	set of natural numbers,
$\mathbb{Z}$	set of integers,
$=$	equal to,
$\neq$	not equal to,
$\exists$	there exists,
$\nexists$	there does not exist,
$\forall$	for all,
$\cup$	union,
$\cap$	intersection,
$\in$	belongs to,
$\notin$	does not belong to,
$\emptyset$	empty set,
$\cong$	isomorphic to,
$\#$	connected sum,
$A \subseteq X$	A is a subset of X,
$A \nsubseteq X$	A is not a subset of X,
$A \times B$	cartesian product of the sets A and B,
$A \setminus B$	set containing elements of A but not B,
$\|V\|$	cardinality of the set V,
$\\|.\\|_e$	Euclidean norm on $\mathbb{R}^m$ for $m \in \mathbb{N}$,
$\lceil \cdot \rceil$	the smallest integer greater than or equal to $(\cdot)$,
$\lfloor \cdot \rfloor$	the largest integer less than or equal to $(\cdot)$,

$[d]$	the set $\{0, 1, \ldots, d\}$,
f_i	i-th coordinate of the f-vector,
g_i	i-th coordinate of the g-vector,
h_i	i-th coordinate of the h-vector,
Δ_S	subcomplex of a balanced simplicial complex Δ corresponding to $S \subseteq [d]$,
β_i	i-th betti number,
$\tilde{\beta}_i$	i-th reduced betti number,
$V(\Delta)$	vertex set of a simplicial complex Δ,
$E(\Delta)$	edge set of a simplicial complex Δ,
$G(\Delta)$	graph of a simplicial complex Δ,
$C(\Delta)$	cone over a simplicial complex Δ,
$\lVert\Delta\rVert$	geometric carrier of a simplicial complex Δ,
$\chi(\Delta)$	Euler characteristic of a simplicial complex Δ,
$m(\Delta)$	rank of the fundamental group of Δ,
$\Delta_1 \star \Delta_2$	join of simplicial complexes Δ_1 and Δ_2,
$\mathrm{lk}_\Delta\sigma$	link of a simplex σ in a simplicial complex Δ,
$\mathrm{st}_\Delta\sigma$	star of a simplex σ in a simplicial complex Δ,
∂P	boundary complex of P, a simplicial polytope,
$\partial\Delta$	boundary complex of Δ, a normal pseudomanifold with boundary,
$\partial\sigma$	boundary complex of a simplex σ,
$\mathbb{S}^n$	n-sphere,
$\mathbb{S}^n_{n+2}$	triangulation of an n-sphere with $n+2$ vertices,
$\mathbb{RP}^n$	n-dimensional real projective space,
$\mathbb{RP}^n_m$	m-vertex triangulation of $\mathbb{RP}^n$,
$\mathbb{CP}^n$	$2n$-dimensional complex projective space,
$\mathbb{CP}^n_m$	m-vertex triangulation of $\mathbb{CP}^n$,
$\mathbb{T}^2$	torus,
$d(\sigma, \Delta)$ (or, $d(\sigma)$)	number of vertices in $\mathrm{lk}_\Delta\sigma$,
$(x, y]$	semi-open and semi-closed edge xy, where $y \in (x, y]$ but $x \notin (x, y]$,
(x, y)	open edge xy, where $x, y \notin (x, y)$.

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