SYNCHRONIZATION CONDITIONS FOR NONLINEAR OSCILLATOR NETWORKS

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JULY 2025

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by

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Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2025

With profound gratitude and deep reverence, I dedicate this thesis to Lord Shiva, whose divine blessings and guidance have given me the strength to endure the trials of this journey. In moments of doubt and exhaustion, it was His grace that kept me moving forward. To my beloved parents, whose love, sacrifices, and unwavering support have been the cornerstone of my life. Their belief in me, even when I doubted myself, has been the greatest source of motivation. Every challenge I overcame and every milestone I achieved is a testament to their endless encouragement and blessings. To my elder brother Rajeev Kumar Pandey, whose wisdom, guidance, and steadfast faith in me have been instrumental in my growth. His constant support and motivation have been my strength, reminding me to stay focused and determined even in the toughest times. This PhD journey has been more than just an academic pursuit; it has been a test of perseverance, resilience, and faith. There were times of struggle, moments of uncertainty, and days when the road ahead seemed impossible. But with the love, blessings, and support of my family and my faith in Lord Shiva, I have reached this milestone. I owe this achievement to them, and with all my heart, I dedicate this work to their unwavering love and belief in me. With deepest gratitude and respect, Sanjeev Kumar Pandey

Certificate

This is to certify that the thesis entitled **Synchronization Conditions For Nonlinear Oscillator Networks**, submitted by **SANJEEV KUMAR PANDEY** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** in Electrical Engineering, is a record of the original, bona fide research work carried out by him under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree.

The results contained in this thesis have not been submitted in part or in full to any other University or Institute for the award of any degree or diploma to the best of our knowledge.

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Acknowledgements

It has been a privilege to study and work under the guidance of **Professor Shaunak Sen** and **Professor Indra Narayan Kar**. I express my deepest gratitude to them for their invaluable insights, continuous encouragement, and unwavering support throughout my journey at IIT Delhi. Beyond academic guidance, I sincerely appreciate their patience and the freedom they provided in my research, which allowed me to explore, learn, and grow. Their mentorship has been instrumental in shaping this thesis.

I am also profoundly grateful to my SRC members, **Professor S. Janardhanan**, **Professor Deepak Umakant Patil**, and **Professor Punit Sharma**, for their insightful feedback and constant support, which played a crucial role in refining my research. Their valuable discussions and willingness to help at every stage of this work were indispensable.

A special note of appreciation goes to Professor **Tapan Kumar Gandhi**, **Professor A. Dewan**, and **Professor Santanu Kapat**, whose exceptional courses provided the strong academic foundation upon which this thesis is built. Their engaging lectures and profound knowledge deeply influenced my research, shaping my understanding and making this journey more meaningful.

The inspiration to persevere and work diligently came from **Professor Bijaya Ketan Panigrahi** (IIT Delhi), one of the most remarkable individuals I have had the privilege of knowing. His encouragement and unwavering belief in my potential instilled the confidence I needed to embark on this journey. I had the honor of publishing my first journal paper under his guidance—an achievement that would not have been possible without his mentorship and trust in my abilities. His wisdom, humility, and compassionate guidance have left a lasting impact on both my academic and personal growth, for which I am deeply grateful.

I would also like to extend my heartfelt thanks to **Professor Samresh Das** (IIT Delhi), whose motivation and support have been instrumental in my academic journey from my master's to my PhD. His encouragement during the challenging phases of my research provided me with the strength to move forward with confidence.

I am immensely grateful to my friends and fellow research scholars, Shivanagouda Biradar, Riddhi Mohan Bora, Apra Gupta, and Dibyajyoti Mukherjee, who stood by me like pillars of strength. In moments of doubt and struggle, they offered not just solutions but also the reassurance that I was never alone in this journey. Their unwavering support made every obstacle easier to overcome.

I would also like to express my heartfelt appreciation to my seniors—Prof. Abhilash Patel, Prof. Sathnesh Singh, Dr. Bhabani Shankar Dey, Dr. Chimmula Kishore, and Er. Apurva Das—whose guidance, wisdom, and camaraderie played an invaluable role in shaping my research experience. Their encouragement made the path smoother and the journey more fulfilling.

Additionally, I extend my sincere gratitude to my juniors—Neetish Patel, Bhargavi Chaudhary, Chayan Paul, Gunjan, and Vinod—for their camaraderie and support. Their presence added joy to my PhD experience, and the moments we shared will always hold a special place in my heart.

I also wish to acknowledge the indispensable contributions of Mr. Virender Singh, Mr. Kailash Rana, and Mr. Sunil Kumar, whose support was crucial in every aspect of my research work. Their dedication and expertise in handling laboratory tasks, assisting with circuit design components, and ensuring a smooth workflow in the lab made an immense difference. Their willingness to help at every step, along with their patience and technical proficiency, have been truly invaluable. I am deeply grateful for their generosity and unwavering support.

Lastly, I owe my deepest gratitude to my parents, whose unconditional love, sacrifices, and unwavering faith in me have been my greatest source of strength. Their constant encouragement and emotional support have given me the resilience to overcome every obstacle along this journey. My heartfelt thanks also go to my elder brother, Senior Engineer Rajeev Kumar Pandey, whose belief in my abilities has always been a driving force behind my perseverance. Their love and encouragement have been the foundation of my success, and I dedicate this achievement to them.

This thesis is the result of the collective efforts of many individuals, and I am truly grateful for each and every one of them.

This journey has been nothing short of extraordinary, and I am profoundly grateful to everyone who has been a part of it.

Abstract

The synchronization of nonlinear oscillators is a fundamental phenomenon with applications across various domains, including neuroscience, power grids, and communication networks. Understanding the conditions that ensure synchronization in networks of coupled oscillators is crucial for the design and control of such complex systems. This thesis addresses the synchronization of nonlinear oscillators under different coupling configurations, establishing theoretical conditions and validating them through simulations and experiments. Using the Lyapunov-Floquet theory and the Master Stability Function (MSF) approach, this study derives necessary and sufficient conditions for synchronization. The analysis is performed for different coupling scenarios, including identical coupling, where a positive coupling constant guarantees synchronization in a network of identical nonlinear oscillators coupled linearly in a full-state fashion; partial-state coupling, where stability is determined through the time evolution of the determinant of the state transition matrix and the behavior of Floquet multipliers; non-identical coupling over an undirected graph for identical nonlinear oscillators, where synchronization conditions are derived based on the spectral properties of the coupling matrix; and non-identical coupling, where synchronization conditions for coupled nonlinear oscillators with non-identical coupling gains over directed graphs are investigated.

The contributions of this work include a comprehensive theoretical framework for synchronization in networks of coupled nonlinear oscillators, proof that positive coupling strength stabilizes eigenmodes in both full-state and partial-state coupling scenarios, validation of theoretical findings through numerical simulations, circuit simulations, and experimental implementations using electronic circuits, and insights into remotely synchronized identical nonlinear oscillators, emphasizing the critical role of coupling strength in maintaining synchronization. The findings in this thesis significantly enhance the understanding of nonlinear oscillator synchronization and provide practical methodologies for designing stable networked systems.

Keywords: Synchronization, Nonlinear Oscillator, Identical Coupling, Non-Identical Coupling, Floquet Theory, Master Stability Function, Remote Synchronization.

सार

गैर-रेखीय दोलकों का समकालन एक मौलिक घटना है, जिसका अनुप्रयोग तंत्रिका विज्ञान, पावर ग्रिड्स और संचार नेटवर्क जैसे विभिन्न क्षेत्रों में होता है। कपल्ड दोलकों के नेटवर्क में समकालन स्थापित होने की अवस्थाओं को समझना, ऐसी जटिल प्रणालियों के डिज़ाइन और नियंत्रण के लिए अत्यंत महत्वपूर्ण है। यह शोध प्रबंध विभिन्न कपलिंग विन्यासों के अंतर्गत गैर-रेखीय दोलकों के समकालन की सैद्धांतिक शर्तों को स्थापित करता है और उनका सत्यापन सिमुलेशन तथा प्रयोगों के माध्यम से करता है। ल्यापुनोव-फ्लोकेट सिद्धांत और मास्टर स्टेबिलिटी फंक्शन (MSF) दृष्टिकोण का उपयोग करते हुए, अध्ययन में समकालन के लिए आवश्यक तथा पर्याप्त शर्तों का निर्धारण किया गया है। विश्लेषण विभिन्न कपलिंग परिदृश्यों के लिए किया गया है, जिसमें समान कपलिंग शामिल है—जहाँ एक धनात्मक कपलिंग स्थिरांक रैखिक रूप से पूर्ण-राज्य में जुड़े हुए एक जैसे गैर-रेखीय दोलकों के नेटवर्क में समकालन की गारंटी देता है; आंशिक-राज्य कपलिंग, जहाँ स्थिरता का निर्धारण राज्य संक्रमण मैट्रिक्स के डेटर्मिनेंट के समयानुसार व्यवहार और फ्लोकेट मल्टीप्लायर्स के अध्ययन द्वारा किया जाता है; अनिर्दिष्ट ग्राफ पर असमान कपलिंग के साथ समान गैर-रेखीय दोलक, जहाँ कपलिंग मैट्रिक्स के स्पेक्ट्रल गुणों के आधार पर समकालन की शर्तें निकाली गई हैं; और निर्देशित ग्राफ पर असमान कपलिंग गेन के साथ कपल्ड गैर-रेखीय दोलक, जिनके लिए समकालन की अवस्थाओं की जांच की गई है।

इस शोध का योगदान कपल्ड गैर-रेखीय दोलकों के नेटवर्क में समकालन हेतु एक समग्र सैद्धांतिक ढांचा प्रस्तुत करना, यह प्रदर्शित करना कि धनात्मक कपलिंग शक्ति पूर्ण-राज्य एवं आंशिक-राज्य दोनों कपलिंग परिदृश्यों में आइगनमोड्स को स्थिर करती है, तथा सैद्धांतिक निष्कर्षों का संख्यात्मक सिमुलेशन, परिपथ सिमुलेशन और इलेक्ट्रॉनिक परिपथों के प्रयोगात्मक कार्यान्वयन द्वारा सत्यापन करना है। इसके अतिरिक्त, दूरस्थ रूप से समकालित समान गैर-रेखीय दोलकों के लिए भी नए दृष्टिकोण शामिल किए गए हैं, जिसमें समकालन बनाए रखने में कपलिंग शक्ति की महत्वपूर्ण भूमिका पर बल दिया गया है। इस प्रबंध के निष्कर्ष गैर-रेखीय दोलक समकालन की समझ को महत्वपूर्ण रूप से बढ़ाते हैं और स्थिर नेटवर्क प्रणालियों के डिज़ाइन के लिए व्यावहारिक दृष्टिकोण उपलब्ध कराते हैं।

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Abbreviations

Symbol	Description
PS	Phase Synchronization
MSF	Master Stability Function
LFT	Lyapunov-Floquet Transformation
LC	Limit Cycle
AJL	Abel-Jacobi-Liouville
LTV	Linear Time-Varying
RS	Remote Synchronization
CS	Complete Synchronization

Symbols

Symbol	Description
$\otimes$	Kronecker Product
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^{m \times n}$	Space of $m \times n$ real matrices
$\mathbb{C}$	Set of complex numbers
$\mathbb{C}^-$	Complex left half of the plane
$\exists$	There exists
$\forall$	For all
$\lambda_{\min}(\mathbf{P})$	Minimum eigenvalue of the matrix $\mathbf{P}$
$\lambda_{\max}(\mathbf{P})$	Maximum eigenvalue of the matrix $\mathbf{P}$
$\lvert\mathcal{N}\rvert$	Cardinality of the set $\mathcal{N}$
$\mathbf{I}_n$	Identity matrix with dimension $n \times n$
$\mathrm{diag}(\cdot)$	A diagonal matrix
$\deg(\cdot)$	Degree of a polynomial
$\det(\cdot)$	Determinant of a matrix
$\mathrm{eig}(\cdot)$	Eigenvalues of a matrix
$\rho(\cdot)$	Spectral radius of a matrix
$\phi(\cdot)$	State Transition Matrix
$\rho(\cdot)$	Floquet Exponent
$\mu_i(\cdot)$	Floquet multiplier
$K(\cdot)$	Coupling Gain