TOPICS IN NUMERICAL SEMIGROUPS

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TOPICS IN NUMERICAL SEMIGROUPS

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Dedicated to

my parents who helped me in all my achievements.

Certificate

This is to certify that the thesis entitled **"Topics in Numerical Semigroups"** submitted by **Edgar Federico Elizeche Armoa** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

New Delhi
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Edgar Elizeche

Edgar Federico Elizeche Armoa

Abstract

In the first chapter, we introduce the theory of numerical semigroups and, in particular, define the Frobenius number and other semigroup invariants. We determine the Frobenius number for numerical semigroups generated by Pythagorean triplets.

In the second chapter, we deal with the problem of invariance of the Frobenius number under deletion of one or more elements from the generating set. We solve the problem for modified arithmetic progressions and compound sequences.

In the third chapter we deal with unique factorization of numerical semigroups generated by compound sequences.

In the fourth chapter, we study fundamental gaps and determine the set of fundamental gaps for numerical semigroups of embedding dimension 2 and for some numerical semigroups of embedding dimension 3, as well as the numerical semigroups generated by arithmetic progressions. We introduce generalized fundamental gaps and close the chapter by calculating the generalized fundamental gaps of order λ for a numerical semigroup of embedding dimension 2.

In the fifth and final chapter, we study the inverse problem for proportionally modular numerical semigroups. The two sections of this chapter deal with the inverse problem for numerical semigroups generated by arithmetic progressions and for numerical semigroups with embedding dimension 3.

सार

पहले अध्याय में, हम न्यूमेरिकल सेमिग्रुप्स की थ्योरी को प्रस्तुत करते हैं और, विशेष रूप से, फ्रोबेनियस नंबर और अन्य सेमिग्रुप इंवरिएंट्स को परिभाषित करते हैं। हम पाइथागोरस ट्रिपलेट्स द्वारा उत्पन्न न्यूमेरिकल सेमिग्रुप्स के लिए फ्रोबेनियस नंबर की गणना करते हैं।

दूसरे अध्याय में, हम जनरेटिंग सेट से एक या अधिक एलिमेंट्स को हटाने पर प्राप्त फ्रोबेनियस नंबर के अपरिवर्तनीय होने की प्रॉब्लम का अध्ययन करते हैं। हम संशोधित अरिथमेटिक प्रोग्रेशन्स और कंपाउंड सीक़्वेन्सेस के लिए यह प्रॉब्लम हल करते हैं।

तीसरे अध्याय में, हम कंपाउंड सीक़्वेन्सेस द्वारा उत्पन्न न्यूमेरिकल सेमिग्रुप्स के अद्वितीय गुणनखंड का अध्ययन करते हैं।

चौथे अध्याय में, हम फंडामेंटल गैप्स का अध्ययन करते हैं, और एम्बेडिंग आयाम 2 वाले न्यूमेरिकल सेमिग्रुप्स के लिए और एम्बेडिंग आयाम 3 वाले कुछ न्यूमेरिकल सेमिग्रुप्स तथा अरिथमेटिक प्रोग्रेशन्स द्वारा उत्पन्न न्यूमेरिकल सेमिग्रुप्स के लिए फंडामेंटल गैप्स के समुच्चय को प्राप्त करते हैं। हम जनरलाइज़्ड फंडामेंटल गैप्स का परिचय देते हैं और एम्बेडिंग आयाम 2 वाले न्यूमेरिकल सेमिग्रुप्स के लिए ऑर्डर λ के जनरलाइज़्ड फंडामेंटल गैप्स की गणना करके अध्याय को समाप्त करते हैं।

पांचवें और अंतिम अध्याय में, हम प्रोपोरशनली मॉड्यूलर न्यूमेरिकल सेमिग्रुप्स के लिए इनवर्स प्रॉब्लम का अध्ययन करते हैं। इस अध्याय के दो खंड, अरिथमेटिक प्रोग्रेशन्स द्वारा उत्पन्न न्यूमेरिकल सेमिग्रुप्स और एम्बेडिंग आयाम 3 वाले न्यूमेरिकल सेमिग्रुप्स के लिए इनवर्स प्रॉब्लम, से संबंधित हैं।

Contents

List of Symbols

Symbol	**Meaning**
S	Numerical Semigroup
$\langle a_1, a_2, \ldots, a_e \rangle$	Numerical semigroup generated by $a_1, a_2, \ldots, a_e$
$e(S)$	Embedding dimension of S
$m(S)$	Multiplicity of S
$G(S)$	Gap set of S
$F(S)$	Frobenius number of S
$g(S)$	Genus of S
$Ap(S, a)$	Apery set of S modulo a
$PF(S)$	Set of pseudo-Frobenius numbers
$t(S)$	Type of S
$FG(S)$	Set of fundamental gaps
$FG(S, \lambda)$	Set of fundamental gaps of order λ
$S(a, b, c)$	Proportionally modular numerical semigroup with factor a, modulus b and proportion c
$S(I)$	Numerical semigroup associated with the interval I

List of Figures

Introduction

The main book regarding the subject is Numerical Semigroups by Rosales and García-Sánchez [34]. One can review this book for more references on the topic of Numerical Semigroups.

Let $\mathbb{N}$ denote the set of nonnegative integers. A *numerical semigroup* S is a subset of $\mathbb{N}$ closed under addition, containing zero and with $|\mathbb{N} \setminus S| < \infty$.

If $a_1, a_2, \ldots, a_e$ are positive integers, then the set $\langle a_1, a_2, \ldots, a_e \rangle = \big\{ \sum_{i=1}^{n} \lambda_i a_i \mid \lambda_i \in \mathbb{N} \big\}$ is the smallest submonoid of $\mathbb{N}$ containing $a_1, a_2, \ldots, a_e$. This submonoid is a numerical semigroup if and only if $\gcd(a_1, a_2, \ldots, a_e) = 1$. In fact, for every numerical semigroup S there exist positive and coprime integers $a_1, a_2, \ldots, a_e$ such that $S = \langle a_1, a_2, \ldots, a_e \rangle$.

The fact that numerical semigroups are interesting stems from the fact that every submonoid of $\mathbb{N}$ is isomorphic to a numerical semigroup.

An interesting question to ask about the numerical semigroup is related to the set of generators, and the following gives us an answer.

Theorem 0.1. *Let S be a numerical semigroup, and let $S^* = S \setminus \{0\}$. Then the set $S^* \setminus (S^* + S^*)$ is the minimal set of generators. Further, any generating set contains the above set.*

The cardinality of the minimal generating set is called the *embedding dimension* of the numerical semigroup. It is usually denoted by $e(S)$. The smallest non-zero element of a numerical semigroup S is called the *multiplicity of S* and is usually denoted by $m(S)$.

One of the most important tools for the study of numerical semigroups is the famous *Apéry Set*, introduced in [1].

Definition 0.2. For any $n \in S$, we define

$$\mathrm{Ap}(S, n) = \Big\{ s \in S \colon s - n \notin S \Big\}.$$

Therefore

$$\mathrm{Ap}(S, n) = \Big\{ w(i) \colon 0 \leq i \leq n-1 \Big\},$$

where $w(i)$ is the least non-negative integer such that $w(i) \equiv i \pmod{n}$.

It is easy to see that $(\mathrm{Ap}(S,n) \setminus \{0\}) \cup \{n\}$ is a generating set for the numerical semigroup S. Therefore, we have that $e(S) \leq m(S)$. In particular, if the equality holds, we say that S has *maximal embedding dimension*.

An important parameter of a numerical semigroup S is its *Frobenius number*, which is the largest number which does not belong to it, sometimes denoted by $F(S)$. Along with the Frobenius number, the *genus*, sometimes denoted by $g(S)$, is another important parameter, defined as the cardinality of $\mathbb{N} \setminus S$.

Among the numbers x that do not belong to the semigroup, the ones that satisfy $x+s \in S^*$ for every $s \in S^*$ are called *pseudo-Frobenius numbers* (note that the Frobenius number satisfies this property, in particular is the biggest number satisfying this property). The set of all pseudo-Frobenius numbers is denoted by $PF(S)$. Its cardinality is important as well, so it gets its own name, the *type* of S, or $t(S)$.

Important formulae to express $F(S)$ and $g(S)$ in terms of the Apéry Set are the following:

Proposition 0.3 ([7, 38])**.** *Let S be a numerical semigroup and $0 \neq n \in S$. Then*

1. $F(S) = \max \mathrm{Ap}(S,n) - n$.
2. $g(S) = \frac{1}{n}\left(\sum_{w \in \mathrm{Ap}(S,n)} w\right) - \frac{n-1}{2}$.

An interesting question is how many numerical semigroups of a given genus exist. Bras-Amorós [5] conjectures that the number of numerical semigroups of a given genus, n_g, grows like the Fibonacci sequence in the sense that $n_{g+1} \geq n_g + n_{g-1}$. So far, even the question of whether n_g is increasing is still open. Zhai [48] proved that this is true for sufficiently large g by proving $\lim_{g\to\infty} \frac{n_{g+1}}{n_g + n_{g-1}} = \varphi$, where φ is the golden ratio. In fact, he proved the stronger result that $\lim_{g\to\infty} n_g \varphi^{-g} = c$ for some constant c. A related question, solved by Eliahou and Ramirez Alfonsín [11], is to determine the number of numerical semigroups generated by two elements of a given genus.

An interesting open problem is Wilf's conjecture [47], that connects the genus, the embedding dimension and the Frobenius number:

$$e(S)g(S) \leq (e(S) - 1)(F(S) + 1).$$

A more recent article by Singhal [39] focuses on the problem of the number of numerical semigroups with a fixed Frobenius number, and analyses the distribution of their genus. Further, he show that if $N(F)$ is the number of numerical semigroups with a fixed Frobenius number, $N(F) < N(F+2)$.

An easier way to calculate the set $PF(S)$ in terms of $\mathrm{Ap}(S,n)$ is the following due to Tripathi [42]:

Proposition 0.4 ([42])**.** *Let A be any set of positive integers with $\gcd(A) = 1$, and let $S = \langle A \rangle$. Let $a \in A$, and let $\mathbf{m}_i$ denote the least integer in S congruent to i modulo a. Then*

$$PF(S) = \left\{ \mathbf{m}_x - a : 1 \leq x \leq a-1 \textit{ and } \mathbf{m}_x + \mathbf{m}_y \geq \mathbf{m}_{x+y} + a \textit{ for } 1 \leq y \leq a-1 \right\}.$$

Note how the $\mathbf{m}_i$ are simply the elements of $\mathrm{Ap}(S, a)$ indexed by the corresponding residue class modulo a.

In the special case where the embedding dimension is 3, studied in [20], Johnson determined that if $S = \langle n_1, n_2, n_3 \rangle$ where $\gcd(n_1, n_2) = \gcd(n_2, n_3) = \gcd(n_1, n_3) = 1$ then one can determine $F(S)$, $g(S)$ and $PF(S)$, by first finding six parameters. We establish this in the following proposition:

Proposition 0.5 ([20])**.** *Let $A = \{n_1, n_2, n_3\}$ be any set of positive integers, such that* $\gcd(n_1, n_2, n_3) = 1$*. Define*

$$c_i = \min\Big\{x \in \mathbb{N}\backslash\{0\} : xn_i \in \langle n_j, n_k \rangle\Big\}$$

or, rewriting it, c_i is the minimum positive integer such that

$$\begin{aligned} c_1 n_1 &= r_{12} n_2 + r_{13} n_3, \\ c_2 n_2 &= r_{21} n_1 + r_{23} n_3, \\ c_3 n_3 &= r_{31} n_1 + r_{32} n_2, \end{aligned}$$

where each $r_{ij} \in \mathbb{N}$. If the elements of A are pairwise coprime, then each $r_{ij} \geq 1$ and each $c_i = r_{ji} + r_{ki}$. Further,

1. $F(S) = c_1 n_1 + \max\{r_{23} n_3, r_{32} n_2\} - (n_1 + n_2 + n_3)$.
2. $g(S) = \frac{1}{2}\Big((c_1 - 1)n_1 + (c_2 - 1)n_2 + (c_3 - 1)n_3 - c_1 c_2 c_3 + 1\Big)$.
3. $PF(S) = \Big\{(c_3 - 1)n_3 + (r_{12} - 1)n_2 - n_1, (c_2 - 1)n_2 + (r_{13} - 1)n_3 - n_1\Big\}$*. In particular,* $t(S) = 2$.

A numerical semigroup S is *irreducible* if S cannot be expressed as the intersection of two numerical semigroups properly containing it, *symmetric* if S is irreducible and $F(S)$ is odd, and *pseudo-symmetric* if S is irreducible and $F(S)$ is even. The definition of irreducibility is not very workable. Luckily for us, we have a more workable characterization.

Lemma 0.6 ([31, Proposition 2])**.** *Let S be a numerical semigroup. Then*

1. *S is symmetric if and only if $PF(S) = \{F(S)\}$.*
2. *S is pseudo-symmetric if and only if $PF(S) = \{F(S), \frac{1}{2}F(S)\}$.*

One can study irreducible numerical semigroups for their applications in Coding theory. For a deeper study of irreducible semigroups, and their applications see [19, 22].

We say that a numerical semigroup S has the *Arf* property if, for any $x, y, z \in S$ with $x \geq y \geq z$, we have $x + y - z \in S$. We say that S is *saturated* if for any $s, s_1, \ldots, s_k \in S$, with $s_i \leq s$ for each $1 \leq i \leq k$, and $n_1, \ldots, n_k \in \mathbb{Z}$ with $\sum n_i s_i \geq 0$, then $s + \sum n_i s_i \in S$. It is evident that every saturated numerical semigroup has the Arf property. What is not so evident, but can be proven, is that every numerical semigroup with the Arf property has maximal embedding dimension, see [34, p. 23].

During the second half of the 20th century, numerical semigroups regain prominence mainly due to their applications in Algebraic Geometry. Valuations of analytically unramified one-dimensional local Noetherian domains are numerical semigroups under certain conditions, and many properties of these rings can be characterized in terms of their associated numerical semigroup. This connection is responsible for many of the terms used in the theory of Numerical Semigroups: multiplicity, embedding dimension, degree of singularity, type, conductor. Some families of numerical semigroups are studied precisely because of this link: numerical semigroups with maximal embedding dimension, symmetric numerical semigroups, pseudo-symmetric numerical semigroups, etc., each have their corresponding concept in Ring Theory. For more references one can check [2].

Recently, the study of certain Diophantine inequalities naturally induced the concept of proportionally modular numerical semigroups whose finite intersections can be linked to the positive cone of certain C^*-algebras.

This thesis is divided into five chapters. In the first chapter we determine the Frobenius number for the family of numerical semigroups generated by a Pythagorean triplet. We also determine all the cases where the numerical semigroup generated by a Pythagorean triplet is irreducible, symmetric, pseudo-symmetric, saturated, or has the Arf property, among others.

In the second chapter we study the problem of characterizing all subsets of a generating set of a numerical semigroup that keep the Frobenius number invariant. We solve this problem for Modified Arithmetic Progressions and Compound sequences.

In the third chapter we deal with unique factorization on the numerical semigroup generated by Compound sequences.

In the fourth chapter, we determine the Fundamental gaps for numerical semigroups with embedding dimension 2 and for some families with embedding dimension 3, as well as for those generated by an arithmetic progression. We close the chapter by introducing the concept of Generalized Fundamental gaps and calculating the Generalized Fundamental gaps of order λ for a numerical semigroup with embedding dimension 2.

In the final chapter we deal with some problems regarding proportionally modular numerical semigroups. The first section of the chapter deals with an inverse problem on numerical semigroups generated by arithmetic progressions. It is well known that a numerical semigroup generated by an arithmetic progression is proportionally modular, i.e., that there exists (i) a triple consisting of a proportion, a modulus and a factor, and (ii) an interval each of which generate the numerical semigroup. Given an numerical semigroup generated by an arithmetic progression, conversely, we characterize the triple of proportion, modulus and factor, and the intervals which generate the numerical semigroup. The second section deals with the analogous problem for numerical semigroups with embedding dimension 3.