EXISTENCE AND REGULARITY RESULTS FOR SOME ELLIPTIC AND PARABOLIC PROBLEMS INVOLVING NON-HOMOGENEOUS OPERATORS

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SEPTEMBER 2021

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by

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Submitted
in fulfillment of the requirements of the degree of
Doctor of Philosophy

to the

Indian Institute of Technology Delhi
September 2021

Dedicated To My Family

Certificate

This is to certify that the thesis entitled **"Existence and Regularity Results for some Elliptic and Parabolic Problems involving Non-homogeneous Operators"** submitted by **Deepak Kumar** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Acknowledgments

With deep sense of salutation and indebtedness, I would like to thank the people without whom this thesis would have been merely a dream.

First and foremost, I would like to express my sincerest gratitude to my advisor, Prof. K. Sreenadh, for his invaluable guidance during my tenure as a researcher at IIT Delhi. His proficiency and expertise in the concerned subject helped me achieve higher productivity as a researcher. He regularly provided an objective analysis and constant motivation, all of which have helped me a lot to sharpen my skills. I thank him for his support and nurturing that has made this thesis possible.

I owe a special thanks to Prof. Jacques Giacomoni (Université de Pau et des Pays de l'Adour, LMAP) for his valuable insights as well as for enlightening me with several ideas. The depth of discussion and collaborative research had always been a tremendous benefit throughout these significant years. I am grateful for the support I received from my collaborator Prof. Vicenţiu D. Rădulescu (University of Craiova, Romania).

I want to thank my SRC (Student Research Committee) members, Dr. Harish Kumar, Dr. Ananta Kumar Majee, and Prof. A. D. Rao for their valuable time and suggestions. As well, the credit for my success belongs to all the teachers who have helped me along the way of my academic journey. I am thankful to CSIR (Council of Scientific and Industrial Research) for the financial support throughout my research work. I would also like to thank Prof. K. Sandeep (TIFR CAM) and Prof. Kapil Sharma (SAU Delhi) for taking time from their busy schedule to attend my CSIR fellowship extension seminars.

Special thanks go to my senior Dr. Divya Goel for providing me with assistance whenever I needed it. I appreciate her constant support and constructive criticism. I am also thankful to my senior Dr. Tuhina Mukherjee for useful suggestions. I wish to thank my best friends and colleagues Vaibhav (NITR), Rakesh (Masaryk), Tanmay (NITJ) and Anshul (IITR) for being with me in every ups and downs of my life.

I would not have enjoyed my stay at IIT Delhi without my friends Vaibhav, Dhiraj, Pradeep, Rohit, Deepak, Navnit, Sandeep, Abhay and Priyamvada. I also thank my juniors Sushmita and Gurdev for their love.

It is with great gratitude that I acknowledge my parents Murlidhar Swarnakar and Anita Kumari for their continuous support and encouragement. This journey would have been impossible without their blessings and sacrifices. I also acknowledge my siblings Vivek and Rupam for their unconditional love.

Finally, I would like to thank the Almighty who has blessed me with an abundance of opportunity and good fortune.

September 2021 *Deepak Kumar*

Abstract

In this thesis, we are concerned with the study of the existence and regularity results of weak solutions to a class of elliptic and parabolic equations driven by the non-homogeneous types of operators, such as the sum of two differential operators with different homogeneity. These kinds of equations have physical applications in biophysics, quantum physics, nonlinear elasticity, chemical reactions and many more. The thesis is divided into five chapters.

In Chapter 1, we present the preliminaries and a brief state of the art regarding the results mentioned in the forthcoming chapters. Additionally, in this chapter, we discuss the organization of the thesis along with the main results of our contribution.

In Chapter 2, we investigate the regularity theory concerning the non-homogeneous quasilinear elliptic problems involving a singular nonlinearity. Particularly, the nonlinear term has the form of a negative power of the unknown function multiplied with a singular weight that blows up near the boundary. We first obtain the existence results by studying the perturbed problem and establishing the behavior of the weak solutions near the boundary. Consequently, we prove the Sobolev regularity and non-existence results. Moreover, we establish $C^{1,\alpha}$ or $C^{0,\alpha}$ regularity results up to the boundary, for some $\alpha \in (0,1)$, of the weak solutions depending on the growth of the nonlinear term. The Hölder continuity result, when the exponent of the nonlinear term is greater than 1, is new even for the homogeneous operator case as it does not require the solution to be in the energy space.

In Chapter 3, we further investigate the regularity theory for strongly non-homogeneous fractional problems. Precisely, we prove the interior Hölder continuity results for local weak solutions to fractional (p,q)-problems, for all $1 < q \leq p < \infty$ and for right hand side in $L^{\gamma}_{\rm loc}$, with suitable γ. Additionally, we establish the Hölder regularity results, up to the boundary, for weak solutions when the right hand side of the equation is bounded. We strengthened these results to the optimal values of the Hölder exponent, for the case of $q \geq 2$. Moreover, we obtain strong maximum and comparison principles. We conclude the chapter by proving Sobolev and Hölder regularity results for a class of doubly singular problem, where singularity occurs due to the negative power of the unknown and a singular weight function that blows up near the boundary.

In Chapter 4, we present the existence and multiplicity results for non-trivial and non-negative weak solutions to fractional (p,q)-problems. Here, we consider the concave-convex nonlinearity having atmost critical growth and sign changing weight functions. Using the method of Nehari manifold and performing some blow up analysis, we establish the

existence of at least two non-negative solutions to the critical exponent problem. Moreover, we obtain the boundedness and Hölder continuity results for the weak solutions.

In Chapter 5, we study a parabolic problem driven by the (p, q)-Laplacian and involving singular nonlinearity with a Carathéodory perturbation having subcritical growth. Here, we first establish several asymptotic properties of the solution to stationary problems through the regularity results obtained in Chapter 2. Subsequently, using time discretization and implicit Euler's method, we prove the global existence result and stabilization property for weak solutions to the parabolic equation when the subcritical perturbation is $(q-1)$-sublinear. Further, when the perturbation is $(q-1)$-superlinear, we obtain the local existence result and prove that the solution exhibits a finite time blowup behavior, which is new even for the homogeneous operator case (that is, $p = q$).

सार

इस थीसिस में हम गैर–सजातीय प्रकार के ऑपरेटरों द्वारा संचालित अण्डाकार और परवलयिक समीकरणों के एक वर्ग के कमजोर समाधानों के अस्तित्व और नियमितता परिणामों का अध्ययन करते हैं, जो कि उदाहरणस्वरुप अलग–अलग समरूपता वाले दो विभेदक ऑपरेटरों के योग के रूप में हो सकते हैं।

प्रथम अध्याय में हम आगामी अध्यायों में उल्लिखित परिणामों के संबंध में मूलभूत संरचना और एक संक्षिप्त सर्वेक्षण प्रस्तुत करते हैं। इसके अतिरिक्त इस अध्याय में हम अगले अध्यायों में प्रस्तुत हमारे योगदान के मुख्य परिणामों के साथ थीसिस के संगठन पर भी चर्चा करते हैं।

द्वितीय अध्याय में हम सिंगुलर गैर–रैखिकता से संबंधित गैर–सजातीय क्वासिलिनियर अण्डाकार समस्याओं के लिए नियमितता सिद्धांत का अध्ययन करते हैं। गैर–रेखीय पद में विशेष रूप से अज्ञात फ़ंक्शन की नकारात्मक शक्ति का रूप होता है जो कि एक विलक्षण भार से गुणा होता है तथा सीमा के पास छितरा जाता है। हम पहले सहायक समस्या का अध्ययन करते हैं और कमजोर समाधानों का सीमा के पास के व्यवहार को स्थापित करके अस्तित्व परिणाम प्राप्त करते हैं। फलस्वरूप हम सोबोलेव नियमितता और गैर–अस्तित्व परिणामों को साबित करते हैं। इसके अलावा, हम गैर–रैखिक पद के विकास के आधार पर कमजोर समाधानों की सार्वत्रिक होल्डर नियमितता की स्थापना करते हैं। होल्डर निरंतरता परिणाम, जब अरेखीय पद का घातांक 1 से अधिक होता है, सजातीय संचालिका मामले के लिए भी नया होता है क्योंकि इसके लिए समाधान की ऊर्जा क्षेत्र में उपस्थिति अनिवार्य नहीं होती है।

तृतीय अध्याय में हम गैर–सजातीय फ्रैक्शनल समस्याओं के लिए नियमितता सिद्धांत की जांच करते हैं। संक्षेप में, हम सभी $1 < q \leq p < \infty$ के लिए फ्रैक्शनल (p, q)–समस्याओं के स्थानीय कमजोर समाधानों के लिए आंतरिक होल्डर निरंतरता परिणाम साबित करते हैं। इसके अतिरिक्त, हम समीकरण के ड़रिचलेट परिसीमा अवस्था के साथ कमजोर समाधानों के लिए सीमा तक होल्डर नियमितता परिणाम स्थापित करते हैं। हमने इन परिणामों को $q \geq 2$ के मामले में होल्डर घातांक के इष्टतम मूल्यों के लिए मजबूत किया। इसके अलावा, हम दृढ़ अधिकतम सिद्धांत और दृढ़ तुलना सिद्धांत प्राप्त करते हैं। हम दोहरी सिंगुलर समस्या के एक वर्ग के लिए सोबोलेव और होल्डर नियमितता को साबित करके अध्याय का समापन करते हैं।

चतुर्थ अध्याय में हम फ्रैक्शनल (p, q)–समस्याओं के गैर–तुच्छ और गैर–ऋणात्मक कमजोर समाधानों के अस्तित्व और बहुलता परिणाम प्रस्तुत करते हैं। यहां हम अवतल–उत्तल गैर–रैखिकता और चिह्न बदलते वजन कार्यों पर विचार करते हैं| नेहारी मैनिफोल्ड की विधि का उपयोग करते हुए और कुछ ब्लो अप विश्लेषण करते हुए, हम क्रिटिकल घातांक समस्या के कम से कम दो गैर–नकारात्मक समाधानों के अस्तित्व को स्थापित करते हैं। इसके अलावा, हम कमजोर समाधानों के लिए सीमाबद्धता और होल्डर निरंतरता परिणाम प्राप्त करते हैं।

पंचम अध्याय में हम (p,q)–लाप्लासियान द्वारा संचालित एक परवलयिक समस्या का अध्ययन करते हैं जिसमे एक सिंगुलर गैर–रैखिकता शामिल होती है। यहां हम अध्याय 2 में प्राप्त नियमितता परिणामों के माध्यम से स्थिर समस्याओं के समाधान के कई स्पर्शोन्मुख गुण स्थापित करते हैं। तत्पश्चात समय पृथक्करण और अंतर्निहित यूलर विधि का उपयोग करके हम परवलयिक समीकरण के कमजोर समाधान के लिए वैश्विक अस्तित्व परिणाम और स्थिरीकरण गुण साबित करते हैं जब सबक्रिटिकल परटरबेशन $(q-1)$–सबलिनीअर है। इसके अलावा जब परटरबेशन $(q-1)$–सुपरलिनीअर होती है, तो हम स्थानीय अस्तित्व परिणाम प्राप्त करते हैं और यह साबित करते हैं कि समाधान एक सीमित समय के पश्चात ब्लो अप व्यवहार प्रदर्शित करता है, जो सजातीय ऑपरेटर मामले के लिए भी नया है।

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List of Symbols

Symbol	Meaning
$\lvert E\rvert$	Lebesgue measure of a set $E \subset \mathbb{R}^N$.
$B_r(x)$	Ball of radius r centered at $x \in \mathbb{R}^N$.
$E \Subset \Omega \subset \mathbb{R}^N$	$\overline{E} \subset \Omega$ and E is bounded.
$⨍_E u(y)dy$	$\frac{1}{\lvert E\rvert}\int_E u(y)dy$.
$(u)_E$	$⨍_E u(y)dy$.
u_+	$\max\{u,0\}$.
u_-	$\max\{-u,0\}$.
1_E	Characteristic function of the set E.
$\mathrm{supp}(u)$	Closure of the set $\{x\ : u(x) \neq 0\}$ in $\mathbb{R}^N$.
$[a]^\ell$	$\lvert a\rvert^{\ell-1}a$, for all $a \in \mathbb{R}$ and $\ell > 0$.
$\mathbb{R}^N_+$	$\{x \in \mathbb{R}^N \mid x_N > 0\}$.
Δ	Laplacian.
Δ_p	$\mathrm{div}(\lvert\nabla u\rvert^{p-2}\nabla u)$, p-Laplacian.
$\lVert f\rVert_{L^p(\Omega)}$	Norm of f in $L^p(\Omega)$.
$[u]_{C^{0,\alpha}(\Omega)}$	$\sup\limits_{x(\neq)y\in\Omega}\dfrac{\lvert u(x)-u(y)\rvert}{\lvert x-y\rvert^\alpha}$, the Hölder seminorm.
$C_c^\infty(\Omega)$	Set of infinitely differentiable functions with compact support in Ω.
$W_0^{1,p}(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$.
$\lVert u\rVert_{W_0^{1,p}(\Omega)}$	$\left(\int_\Omega \lvert\nabla u\rvert^p dx\right)^{1/p}$, norm of u in $W_0^{1,p}(\Omega)$.
p^*	$\frac{Np}{N-p}$ if $N > p$, otherwise an arbitrary large number, the Critical Sobolev exponent.
$\lVert u\rVert_{W^{s,p}(\Omega)}$	Norm of u in the fractional order spaces $W^{s,p}(\Omega)$.
$[u]_{W^{s,p}(\Omega)}$	$\int_\Omega\int_\Omega \frac{\lvert u(x)-u(y)\rvert^p}{\lvert x-y\rvert^{N+ps}}\,dxdy$.
p_s^*	$\frac{Np}{N-ps}$ if $N > ps$, otherwise an arbitrary large number.