ALGORITHMIC AND STRUCTURAL ASPECTS OF MATCHING WITH PREFERENCES

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ALGORITHMIC AND STRUCTURAL ASPECTS OF MATCHING WITH PREFERENCES

by

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Department of Mathematics

Submitted
in fulfillment of the requirements of the degree of
Doctor of Philosophy

to the



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Dedicated

with a heart full of gratitude

to

the person who has shaped me-

My Guru

Professor Bhawani Sankar Panda.

कृतज्ञता से भरे हृदय से

उस शख़्सियत को

समर्पित

जिसने मुझे तराशा है-

मेरे गुरु

आचार्य भवानी शंकर पंडा

Certificate

This is to certify that the thesis entitled **Algorithmic and Structural Aspects of Matching with Preferences**, submitted by **Mr. Sachin**, to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted, in full or in parts, to any other university or institute for the award of any degree or diploma.

New Delhi
July 2026

Prof. Bhawani Sankar Panda
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Abstract

Given n men, n women, and their preference lists containing all persons of the opposite gender in order of preference, the Stable Marriage problem (SM) asks to match men and women so that no two individuals of opposite gender prefer each other to their current partners, that is, to find a matching that is stable according to the preferences. The Stable Marriage problem, in which the preference lists may contain ties, is denoted as SMT, and the Stable Marriage problem, in which the preference lists may be incomplete, is denoted as SMI. SMTI denotes the Stable Marriage problem, in which preference lists may contain ties and be incomplete. The thesis deals with an algorithmic and complexity-theoretic study of matching problems involving preferences under structural and geometric constraints.

The first part of the thesis introduces three novel variants of SMTI: SMTI-C (preferences are *consecutive*), SMTI-INC (*inclusive*), and SMTI-STEP (*steps-like structure*), motivated by realistic preference patterns arising in admission systems and ranking-based markets. For these variants, we analyze classical decision and optimization problems, including the existence of complete stable matchings and the computation of maximum and minimum cardinality stable matchings. We establish strengthened NP-completeness results, prove W[1]-hardness when parameterized by the total number of ties, and identify polynomial-time solvable cases under additional structural assumptions. Furthermore, we derive improved approximation guarantees

for certain variants, thereby refining the known approximability landscape of SMTI.

The second part of the thesis focuses on stable matchings under geometric restrictions. In particular, we study strongly stable non-crossing matchings when persons are arranged in parallel on line segments. The problem of determining the existence of a strongly stable non-crossing matching is polynomial-time solvable in the classical setting without ties, but is known to be NP-complete in the presence of ties in the lists. We show fixed-parameter tractability with respect to the total length of ties and examine the complexity under alternative geometric layouts, including circular and general 2-dimensional planar arrangements. Additionally, we introduce an alternate stability concept— semi-strongly stable non-crossing matching, and characterize its computational complexity, establishing both hardness results and linear-time solvable instances.

We then turn to the Stable Minimum Crossing Matching (SMCM) problem, which asks for a stable matching that minimizes the total number of crossings. We answer an open problem by showing that SMCM is NP-hard in general, derive lower and upper approximation bounds, and design polynomial-time algorithms for structured instances based on master preference lists. Furthermore, we provide fixed-parameter tractable algorithms parameterized by structural properties of the underlying graph.

Motivated by the limitations on the existence of stable non-crossing matchings, the final part of the thesis investigates relaxed geometric models in which the number of crossings per edge is bounded, and relaxed stability models in which the number of blocking pairs is bounded. In the relaxed geometric model, we analyze the classical and parameterized complexity of the stable matchings that admit at most one, or at most k crossings per edge. In the relaxed stability model, we examine non-crossing matchings that are 'almost' stable, that is, which admit the minimum number of blocking pairs. We present NP-completeness and inapproximability results and discuss the polynomial-time solvable variants of this model.

सारांश

हम संरचनात्मक और ज्यामितीय बाधाओं के तहत प्राथमिकताओं से जुड़ी मिलान समस्याओं का एक एल्गोरिथमिक और जटिलता-सैद्धांतिक अध्ययन प्रस्तुत करते हैं। दिए गए *क* पुरुषों, *क* महिलाओं और प्राथमिकता के क्रम में विपरीत लिंग के व्यक्तियों वाली उनकी प्राथमिकता सूचियों को देखते हुए, स्थिर विवाह समस्या (एस.एम.) पुरुषों और महिलाओं का मिलान करने के लिए कहती है ताकि विपरीत लिंग के कोई भी दो व्यक्ति एक-दूसरे को अपने वर्तमान साथी से अधिक पसंद न करें, अर्थात् एक ऐसा मिलान जो प्राथमिकताओं के अनुसार स्थिर हो। एस.एम.टी. उस स्थिर विवाह समस्या को दर्शाता है जिसमें प्राथमिकता सूचियों में व्यक्ति समान, अर्थात् बंधे हुए (टाई) हो सकते हैं तथा एस.एम.आई. उस स्थिर विवाह समस्या को दर्शाता है जिसमें प्राथमिकता सूचियाँ अपूर्ण (इन्कम्प्लीट) हो सकती हैं। एस.एम.टी.आई. उस स्थिर विवाह समस्या को दर्शाता है जिसमें प्राथमिकता सूचियों में व्यक्ति समान, अर्थात् बंधे हुए (टाई) हो सकते हैं तथा प्राथमिकता सूचियाँ अपूर्ण (इन्कम्प्लीट) हो सकती हैं।

शोध प्रबंध का पहला भाग एस.एम.टी.आई. के तीन नवीन रूपों का परिचय देता हैः एस.एम.टी.आई-सी (प्राथमिकताएं लगातार हैं), एस.एम.टी.आई-आईएनसी (समावेशी प्राथमिकताएं) और एस.एम.टी.आई-स्टेप (कदम जैसी संरचना) जो प्रवेश प्रणालियों और रैंकिंग-आधारित बाजारों में उत्पन्न होने वाले यथार्थवादी प्राथमिकता पैटर्न से प्रेरित हैं। इन रूपों के लिए, हम शास्त्रीय निर्णय और अनुकूलन समस्याओं का विश्लेषण करते हैं, जिसमें पूर्ण स्थिर मिलान के अस्तित्व और अधिकतम तथा न्यूनतम आकार वाले स्थिर मिलान की गणना शामिल है। हम एनपी-पूर्णता परिणामों को मजबूत करते हैं, कुल बंधनों की संख्या द्वारा मानकीकृत डब्ल्यू[१]-कठिनता को साबित करते हैं, और अतिरिक्त संरचनात्मक मान्यताओं के तहत बहुपद-समय में हल होने योग्य मामलों की पहचान करते हैं। इसके अलावा, हम कुछ रूपों के लिए बेहतर सन्निकटन गारंटी प्राप्त करते हैं, जिससे एस.एम.टी.आई. के ज्ञात सन्निकटन परिदृश्य को परिष्कृत किया जाता है।

शोध प्रबंध का दूसरा भाग ज्यामितीय प्रतिबंधों के तहत स्थिर मिलानों पर केंद्रित है। विशेष रूप से, हम रेखा खंडों पर समांतर रूप से व्यवस्थित व्यक्तियों के लिए प्रबल स्थिर गैर-अद्रुवी (नॉन-क्रासिंग) मिलानों का अध्ययन करते हैं। प्रबल स्थिर गैर-अद्रुवी मिलान के अस्तित्व का निर्धारण बिना बंधनों के शास्त्रीय व्यवस्था में बहुपद-समय में हल होने योग्य है, लेकिन सूचियों में बंधनों की उपस्थिति में एनपी-पूर्ण ज्ञात है। हम कुल बंधनों की लंबाई के सापेक्ष फिक्स्ड-पैरामीटर ट्रैक्टेबिलिटी दिखाते हैं और

वैकल्पिक ज्यामितीय विन्यासों के तहत जटिलता की जाँच करते हैं, जिसमें वृत्तीय और सामान्य २-आयामी समतलीय व्यवस्थाएँ शामिल हैं। इसके अतिरिक्त, हम एक वैकल्पिक स्थिरता अवधारणा - अर्ध-प्रबल स्थिर गैर-अद्रुवी मिलान, का परिचय देते हैं, इसकी गणनात्मक जटिलता को विशेषित करते हैं, और कठिनता परिणामों तथा रैखिक-समय में हल होने वाले मामलों को स्थापित करते हैं।

इसके उपरांत हम स्थिर न्यूनतम अद्रुवी मिलान (एस.एम.सी.एम.) समस्या की ओर रुख करते हैं, जो कुल अद्रुवी संख्या को न्यूनतम करने वाले स्थिर मिलान को ढूंढने का प्रश्न करती है। हम एक अनसुलझी समस्या को हल करते हैं और दिखाते हैं कि एस.एम.सी.एम. सामान्य रूप से एन.पी.-कठिन है। हम इसकी निचली और ऊपरी सन्निकटन सीमाएं प्राप्त करते हैं, और मास्टर प्राथमिकता सूचियों पर आधारित संरचित उदाहरणों के लिए बहुपद-समय एल्गोरिदम बनाते हैं। इसके अलावा, हम अंतर्निहित ग्राफ के संरचनात्मक गुणों द्वारा मानकीकृत फिक्स्ड-पैरामीटर ट्रैक्टेबल एल्गोरिदम प्रदान करते हैं।

स्थिर गैर-अद्रुवी मिलानों के अस्तित्व की सीमाओं से प्रेरित होकर, शोध प्रबंध का अंतिम भाग उन आरक्षित ज्यामितीय मॉडलों जिनमें प्रति किनारे अद्रुवी की संख्या सीमित है, और उन आरक्षित स्थिरता मॉडलों जिनमें अवरोधक जोड़ों की संख्या सीमित है, की जाँच करता है। आरक्षित ज्यामितीय मॉडल में, हम प्रति किनारे अधिकतम एक या अधिकतम k अद्रुवी वाले स्थिर मिलानों की शास्त्रीय और मानकीकृत जटिलता का विश्लेषण करते हैं। आरक्षित स्थिरता मॉडल में, हम लगभग स्थिर, अर्थात् न्यूनतम अवरोधक जोड़ों वाले गैर-अद्रुवी मिलान की जाँच करते हैं। हम एनपी-पूर्णता और असन्निकट-योग्यता परिणाम प्रस्तुत करते हैं और इस मॉडल के बहुपद-समय में हल होने वाले रूपों पर चर्चा करते हैं।

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List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	Set of *natural numbers*
$\mathbb{Z}$	Set of *integers*
$\forall\, x$	For *all* x
$x \in X$	x is a *member* of X
$A \subseteq X$	A is a *subset* of X
$\lvert X \rvert$	The *cardinality* of a set X
$A \cap B$	The *intersection* of A and B
$A \cup B$	The *union* of A and B
$A \setminus B$	The *set difference* of A and B
$A \triangle B$	The *symmetric difference* of A and B
$A \times B$	The *Cartesian product* of A and B
$\emptyset$	The *empty* set
$\square$	The *end of a proof*
$\mid$	such that
$E(G)$	The set of *edges* of G
$V(G)$	The set of *vertices* of G
$d(v)$	The *degree* of the vertex v

Δ or $\Delta(G)$	The *maximum degree* in G
δ or $\delta(G)$	The *minimum degree* in G
$N(v)$	*neighborhood* of v in G
$G[H]$	The subgraph of G *induced* by the vertices of H
$G = (X, Y, E)$	The *bipartite graph* G with vertex set $X \cup Y$
$G = (X \cup Y, E)$	The *bipartite graph* G with vertex set $X \cup Y$
C_n	The *cycle* on n vertices
P_n	The *path* on n vertices
P	The class of *deterministic* polynomial-time solvable problems
NP	The class of *nondeterministic* polynomial-time solvable problems