COMPUTATION OF OPTIMAL CONSENSUS IN MULTI AGENT SYSTEMS USING GRÖBNER BASIS

AKANSHA RAUTELA



DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2026

© Indian Institute of Technology Delhi (IITD), New Delhi, 2026

Computation of Optimal Consensus in Multi Agent Systems Using Gröbner Basis

by

AKANSHA RAUTELA

Department of Electrical Engineering

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2026

Dedicated to my grandfather, family & friends !

Certificate

This is to certify that the thesis entitled **Computation of Optimal Consensus in Multi Agent Systems Using Gröbner Basis**, submitted by **Akansha Rautela** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** in Electrical Engineering, is a record of the original, bona fide research work carried out by him under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree.

The results contained in this thesis have not been submitted in part or in full to any other University or Institute for the award of any degree or diploma to the best of our knowledge.

Prof. Deepak U.Patil
Department of Electrical Engineering,
Indian Institute of Technology Delhi.
(Supervisor)

Prof. Indra Narayan Kar
Department of Electrical Engineering,
Indian Institute of Technology Delhi.
(Supervisor)

Acknowledgements

I sincerely thank my advisors, *Prof. Deepak U. Patil* and *Prof. Indra Narayan Kar*, for their constant guidance, encouragement, and support throughout my doctoral studies. Their deep insight, positive approach to research, and emphasis on clarity of thought have greatly influenced my understanding of control theory and research as a whole. I am especially grateful for their patience and for continuously encouraging me to explore challenging problems and strive for better outcomes. I am deeply grateful to the members of my research committee, *Prof. S. Janardhanan, Prof. Subashish Datta*, and *Prof. Punit Sharma*, for their valuable feedback and thoughtful suggestions. Their constructive criticism at various stages has significantly helped in improving the quality of this work.

I would also like to acknowledge Dr. Zahir Ahmad Ansari Scientist, IRDE(DRDO), Dehradun, whose mentorship during my Master's internship played an important role in motivating me to pursue research.

I extend my sincere thanks to staff of the Control and Automation group at IIT Delhi Sunil Ji, Virender Ji, Kailash Ji for providing a supportive academic environment and the necessary resources for carrying out this research.

I am grateful to my seniors, Dr. Bhabani, Dr. Rajasree, Dr. Sujeet, Dr. Keerthi, Dr. Midhun, Dr. Shivan, Dr. Eswara, Dr. Krishanu Nath, Pranjali Shukla for their guidance and support during the early stages of my Ph.D. Their experience and willingness to help made it to navigate both academic and practical challenges.

I would like to thank my lab mates and colleagues Neetish, Chayan, Bhargavi, Riddhi, Anchita, Poulomee, Soutrik, Vinod, Gunjan, Rudra Sen, Rudra Prakash, Chaitra for being an integral part of this journey. The many discussions, brain-storming sessions, and shared experiences, both academic and personal, tea breaks made this journey more engaging and enjoyable. Their support and camaraderie created a positive and motivating environment in the lab.

I am also thankful to Shekhar for being a source of support during challenging times. Also Reetam, and Ruchi, for their constant encouragement.

Finally, I owe my deepest gratitude to all my family members, especially my grandfather, parents, and brother, for their unconditional love, understanding, and unwavering belief in me. Their support has been my strongest foundation throughout this journey.

I am also deeply grateful to the campus flora and fauna the quiet companions that provided much-needed serenity, stress relief, and perspective over the years!

Lastly, I also extend my heartfelt thanks to all the friends, colleagues, and well-wishers whose names may not appear here, but whose support was invaluable to the completion of this thesis. :)

Abstract

This thesis presents a distributed framework for computing optimal consensus in multi-agent systems subject to resource constraints, specifically fuel and energy. We consider a group of N identical linear time-invariant agents with n states, bounded control inputs, and formulate the consensus problem as finding the least time when the attainable sets of all agents intersect, and the corresponding consensus state is the point of intersection. This set-theoretic formulation utilizes the geometric properties of attainable sets and naturally leads to a characterization of fundamental limitations on consensus in terms of the agents' initial conditions, available time, and control resources such as fuel and energy. However, as the number of agents increases, direct computation of the global intersection becomes computationally intractable. To address this challenge, Helly's theorem from convex geometry is used to decompose the computation into ${}^{N}C_{n+1}$ subproblems. In each subproblem, the minimum time and the corresponding point at which the intersection of attainable sets for a collection of $(n+1)$ agents becomes non-empty are determined. We develop a systematic and distributed procedure to solve these subproblems and to combine their solutions, to find the minimum time to consensus, at which N agents can reach the corresponding consensus point while respecting the bounded energy constraints. The solution of each subproblem reduces to solving a finite set of polynomial equations, for which Gröbner basis–based elimination method results in closed-form expressions and a computationally tractable characterization of the optimal consensus solution. The proposed framework is developed for agents with second-order dynamics, including double integrators and a class of damped second-order linear time-invariant systems under fuel budget constraints. Further, under the energy budget constraints (as opposed to fuel budget), we are able to generalize the framework for general n^{th} order agent dynamics. In particular, we focus on a)

minimum time consensus with fuel budget constraints, b) minimum fuel budget allocations for consensus at a given time, and c) minimum time consensus with energy budget constraints.

Keywords: Linear systems, bang-off-bang control, resource constrained consensus, fuel optimal control, energy optimal control, multi-agent systems, Gröbner basis, Attainable sets.

सारांश

यह थीसिस संसाधन बाधाओं (विशेष रूप से ईंधन और ऊर्जा) के अधीन मल्टी-एजेंट सिस्टम में इष्टतम सर्वसम्मति (optimal consensus) की गणना के लिए एक वितरित रूपरेखा (distributed framework) प्रस्तुत करती है। हम N समान रैखिक समय-अपरिवर्तनीय (linear time-invariant) एजेंटों के एक समूह पर विचार करते हैं जिनकी n अवस्थाएँ और परिबद्ध नियंत्रण इनपुट (bounded control inputs) हैं। हम सर्वसम्मति समस्या को उस न्यूनतम समय को खोजने के रूप में तैयार करते हैं जब सभी एजेंटों के प्राप्य समुच्चय (attainable sets) प्रतिच्छेद (intersect) करते हैं, और यह प्रतिच्छेदन बिंदु ही संबंधित सर्वसम्मति अवस्था होती है।यह समुच्चय-सैद्धांतिक (set-theoretic) निरूपण प्राप्य समुच्चयों के ज्यामितीय गुणों का उपयोग करता है और स्वाभाविक रूप से एजेंटों की प्रारंभिक स्थितियों, उपलब्ध समय, तथा ईंधन और ऊर्जा जैसे नियंत्रण संसाधनों के संदर्भ में सर्वसम्मति की मूलभूत सीमाओं को स्पष्ट करता है।हालाँकि, जैसे-जैसे एजेंटों की संख्या बढ़ती है, वैश्विक प्रतिच्छेदन (global intersection) की प्रत्यक्ष गणना कम्प्यूटेशनल रूप से अत्यंत जटिल हो जाती है। इस चुनौती को दूर करने के लिए, उत्तल ज्यामिति (convex geometry) से हेली की प्रमेय (Helly's theorem) का उपयोग करके इस गणना को ${}^{N}C_{n+1}$ उपसमस्याओं में विभाजित किया जाता है। प्रत्येक उपसमस्या में, वह न्यूनतम समय और संबंधित बिंदु निर्धारित किया जाता है जिस पर $(n+1)$ एजेंटों के संग्रह के लिए प्राप्य समुच्चयों का प्रतिच्छेदन गैर-रिक्त (non-empty) हो जाता है।हम इन उपसमस्याओं को हल करने और उनके समाधानों को संयोजित करने के लिए एक व्यवस्थित और वितरित प्रक्रिया विकसित करते हैं, ताकि सर्वसम्मति का न्यूनतम समय ज्ञात किया जा सके, जिस पर N एजेंट परिबद्ध ऊर्जा बाधाओं का पालन करते हुए संबंधित सर्वसम्मति बिंदु तक पहुँच सकें। प्रत्येक उपसमस्या का समाधान बहुपद समीकरणों (polynomial equations) के एक सीमित सेट को हल करने तक कम हो जाता है, जिसके लिए ग्रॉबनर बेसिस (Gröbner basis) आधारित विलोपन विधि (elimination method) बंद-रूप व्यंजक (closed-form expressions) और इष्टतम सर्वसम्मति समाधान का एक कम्प्यूटेशनल रूप से सुगम लक्षण वर्णन प्रदान करती है।प्रस्तावित रूपरेखा ईंधन बजट बाधाओं के तहत द्वितीय-क्रम गतिशीलता (second-order dynamics) वाले एजेंटों के लिए विकसित की गई है, जिसमें डबल इंटीग्रेटर और डैम्प्ड द्वितीय-क्रम रैखिक समय-अपरिवर्तनीय प्रणालियों का एक वर्ग शामिल है। इसके अतिरिक्त, ऊर्जा बजट बाधाओं (ईंधन बजट के विपरीत) के तहत, हम सामान्य n^{th} क्रम की एजेंट गतिशीलता के लिए इस रूपरेखा को सामान्यीकृत करने में सक्षम हैं। विशेष रूप से, हम निम्न पर ध्यान केंद्रित करते हैं: a) ईंधन बजट बाधाओं के साथ न्यूनतम समय सर्वसम्मति, b) एक निश्चित समय पर सर्वसम्मति के लिए न्यूनतम ईंधन बजट आवंटन, और c) ऊर्जा बजट बाधाओं के साथ न्यूनतम समय सर्वसम्मति।

मुख्य शब्द (Keywords): **रैखिक प्रणालियाँ** (Linear systems), **बैंग-ऑफ-बैंग नियंत्रण** (bang-off-bang control), **संसाधन विवश सर्वसम्मति** (resource constrained consensus), **ईंधन इष्टतम नियंत्रण** (fuel optimal control), **ऊर्जा इष्टतम नियंत्रण** (energy optimal control), **मल्टी-एजेंट सिस्टम** (multi-agent systems), **ग्रॉबनर बेसिस** (Gröbner basis), **प्राप्य समुच्चय** (Attainable sets)।

Contents

Certificate i

Acknowledgements ii

Abstract iv

Contents ix

List of Figures xiii

List of Tables xv

Abbreviations xvii

Symbols xix

1 Introduction and Organization 1
- 1.1 Background . . . 1
- 1.2 Motivation . . . 2
- 1.3 Optimal Consensus Problem with Resource-Constraints . . . 3
- 1.4 Contributions . . . 8
- 1.5 Organization of the Thesis . . . 10
- 1.6 Outcomes of the Thesis . . . 13

2 Geometry of Attainable and Reachable sets 15
- 2.1 Optimal Resource Constrained Control . . . 16
- 2.2 Attainable Set under Fuel Budget . . . 18
  - 2.2.1 Double integrator system with fuel constraints . . . 21
  - 2.2.2 Damped dynamics with fuel constraints . . . 25
  - 2.2.3 Properties of $\mathcal{A}^{\beta}(t_f, \mathbf{x}_0)$ . . . 30

2.3 Attainable set under energy budget . . . 32
2.4 Summary . . . 37

3 Minimum Time Consensus for Double Integrator Agents 39
3.1 Introduction . . . 39
3.2 Problem Formulation . . . 41
3.3 Consensus Region . . . 42
3.4 Minimum Time Consensus for a Triplet of Agents . . . 44
3.4.1 Reducing the number of possibilities . . . 46
3.4.1.1 Invalid possibilities due to Assumption 3.1 . . . 46
3.4.1.2 Invalid possibilities due to non-singleton intersection 47
3.4.1.3 Remaining 4 invalid possibilities . . . 49
3.4.1.4 Valid possibilities . . . 49
3.4.2 $\partial\mathcal{A}_i^{\beta,s_1} \cap \partial\mathcal{A}_j^{\beta,s_3} \cap \partial\mathcal{A}_k^{\beta,s_1}$ . . . 50
3.4.3 $\partial\mathcal{A}_i^{\beta,s_3} \cap \partial\mathcal{A}_j^{\beta,s_2} \cap \partial\mathcal{A}_k^{\beta,s_1}$ . . . 51
3.5 Minimum Time Consensus for N Agents . . . 52
3.5.1 Distributed Computation . . . 54
3.5.2 Computational Complexity . . . 56
3.6 Example . . . 58
3.7 Summary . . . 60

4 Minimum Time Consensus for Damped Second-order Agents 61
4.1 Introduction . . . 61
4.2 Problem Formulation . . . 63
4.3 Region of Consensus . . . 64
4.4 Minimum Time Consensus . . . 66
4.4.1 Minimum Time Consensus for a Triplet of Agents . . . 66
4.4.1.1 $\partial\mathcal{A}_i^{\beta,s_1} \cap \partial\mathcal{A}_j^{\beta,s_3} \cap \partial\mathcal{A}_k^{\beta,s_1}$ . . . 68
4.4.1.2 $\partial\mathcal{A}_i^{\beta_2,s_3} \cap \partial\mathcal{A}_j^{\beta_2,s_2} \cap \partial\mathcal{A}_k^{\beta_2,s_1}$ . . . 70
4.4.2 Minimum Time Consensus for N Agents . . . 70
4.5 Example . . . 72
4.6 Summary . . . 74

5 Minimum Fuel Budget Allocations for Consensus 75
5.1 Introduction . . . 75
5.2 Problem Formulation . . . 76
5.3 Bounds on Minimum Total Fuel . . . 78
5.3.1 Lower Bound . . . 79
5.3.2 Upper Bound . . . 80
5.3.3 Characterization of Initial Condition for Equal Bounds . . . 83
5.4 Computing UMFB $\bar{\beta}$ . . . 84
5.4.1 Computing $\bar{\beta}_{ijk}$ for triplet $\{a_i, a_j, a_k\}$ . . . 85

5.4.1.1 $\partial\mathcal{A}_i^{\hat{\beta}_1,s_1} \cap \partial\mathcal{A}_j^{\hat{\beta}_1,s_3} \cap \partial\mathcal{A}_k^{\hat{\beta}_1,s_1}$ . . . 88
5.4.1.2 $\partial\mathcal{A}_i^{\hat{\beta}_2,s_3} \cap \partial\mathcal{A}_j^{\hat{\beta}_2,s_2} \cap \partial\mathcal{A}_k^{\hat{\beta}_2,s_3}$ . . . 90
5.4.2 Computing $\bar{\beta}$ for N-agents . . . 92
5.5 Examples . . . 94
5.5.1 Example 1: The upper and lower bounds are unequal . . . 94
5.5.2 Example 2: The upper and lower bounds are equal . . . 96
5.6 Summary . . . 97

6 Minimum time consensus with energy constraints 99
6.1 Introduction . . . 99
6.2 Problem Formulation . . . 101
6.3 Minimum time consensus computation . . . 104
6.3.1 Minimum time consensus for $n+1$ agents in $\mathcal{L}_\nu$ . . . 104
6.3.1.1 Cases yielding a nonempty intersection: . . . 105
6.3.2 Computing the intersection of m agents in $\mathcal{L}_\nu$ . . . 106
6.3.3 Solution using Gröbner basis . . . 107
6.3.3.1 **Case 1:** All eigenvalues are real rational and distinct 107
6.3.3.2 **Case 2**: $2m$ eigenvalues are complex conjugate pairs and remaining $n-2m$ are real and rational . . . 108
6.3.3.3 **Case 3:** Repeated real rational eigenvalues . . . 109
6.4 Minimum Time Consensus for all N Agents . . . 113
6.5 Examples . . . 114
6.6 Alternative heuristic approach to avoid Gröbner basis computation . 117
6.7 Summary . . . 120

7 Conclusions and Future Directions 123
7.1 Concluding Remarks . . . 123
7.2 Future Scopes . . . 126

A Gröbner Basis Preliminaries 137

B Proof of Helly's Theorem 141

C Code Repository 147

References 149

Bio-data 151

List of Figures

1.1 Altitude alignment of multiple drones 2
1.2 Thesis organization 12

2.1 Time optimal Attainable set from $\mathbf{x}_0 = [0,0]^\top$ with $s_1 = \{+1,-1\}$ and $s_3 = \{-1,+1\}$ sequences. 20
2.2 $\partial\mathcal{A}^\beta(t_f,\mathbf{x}_0)$ with $\mathbf{x}_0 = [0,0]^\top$, $\beta = 50$ 25
2.3 Attainable set for $\beta = 0.7$ and $b = 1$ 29
2.4 Inclusion property of $\mathcal{A}^\beta(t_f,\mathbf{x}_0)$ w.r.t. β for $\mathbf{x}_0 = [0,0]^\top$ 31
2.5 Attainable set for energy-constrained input with $\beta = 0.7$ 37

3.1 Consensus region for $N = 12$ agents 43
3.2 Two cases for the intersection of attainable set for triplet $\{a_i, a_j, a_k\}$. 45
3.3 Invalid case violating the Assumption 3.1 47
3.4 Invalid case due to non-singleton intersection 48
3.5 Invalid case due to the same boundaries intersecting, leading to a non-singleton intersection 49
3.6 $\partial\mathcal{A}_i^\beta, \partial\mathcal{A}_j^\beta, \partial\mathcal{A}_k^\beta$ with $\mathbf{x}_{i0} = [0,0]^\top$,$\mathbf{x}_{j0} = [40,64]^\top$, $\mathbf{x}_{k0} = [-1500,25]^\top$, $\beta = 50$, $\bar{\mathbf{x}}_{ijk} = [3236.5, 32.37]^\top$. 51
3.7 Convex hull of initial conditions inside reachable set 58
3.8 State trajectories and attainable sets for $N = 12$ agents. 60

4.1 Region of Consensus 65
4.2 Intersection of Attainable sets for a triplet of agents 69
4.3 State trajectories and attainable sets for $N = 12$ agents. 73

5.1 $\mathcal{R}^{20}(100,\bar{\mathbf{x}}) \subseteq \mathcal{R}^{40}(100,\bar{\mathbf{x}}) \subseteq \mathcal{R}^{60}(100,\bar{\mathbf{x}})$ with $\bar{\mathbf{x}} = [2000,10]^\top$ 84
5.2 Two possible cases for consensus among agents 88
5.3 Phase plots, and attainable sets for $N = 6$ agents 95
5.4 Bang-off-bang control profile 95
5.5 System response for $N = 6$ agent system under the condition where upper and lower bounds meets. 97
5.6 Bang-off-bang control profile 97

6.1 Variation of $\partial\mathcal{A}^\beta(t_f,\mathbf{x}_0)$ with t_f for $\beta = 2$ and $\mathbf{x}_0 = [1,1]^\top$ 103

6.2 $\cap_{i=1}^{3}\partial\mathcal{A}_i^{\beta}(t_f, \mathbf{x}_{i0})$ with $\beta = 0.7$ and $t_f = 2.09$ 112
6.3 State-trajectories and attainable set intersection for $N = 6$ agents . . 115
6.4 Control profile 116
6.5 State trajectories 119
6.6 Minimum energy inputs to reach the optimal consensus point 119
6.7 Attainable set plots 120

List of Tables

1.1 Key differences between Energy and Fuel Costs in Optimal Control . 6

3.1 Control inputs $u_i(t)$ and fuel used by agent a_i 59

4.1 Fuel utilization and control profiles of $N = 12$ agents to reach $\bar{\mathbf{x}} = [0.2781, 0.0941]^\top$ at the minimum time $\bar{t}_f = 1.4918$. 74

5.1 Parameters for agents $\{a_1, a_4, a_6\}$ with final time $t_f = 100$. 95
5.2 Updated information for remaining agents. 96
5.3 Details of $N = 6$ agents satisfying the equality conditions for the upper and lower bounds. 96

6.1 Energy consumption per agent at $\bar{t}_f = 8.1586$. 115
6.2 Energy spent by agents to reach $\bar{\mathbf{x}} = [0.0443, -0.3218, 0.2123]^\top$ at $\bar{t}_f = 2.6101$. 118

Abbreviations

LTI	**L**inear **T**ime **I**nvariant
PMP	**P**ontryagin's **M**inimum **P**rinciple
UMFB	**U**niform **M**inimum **F**uel **B**ound
UAVs	**U**nmanned **A**erial **V**ehicles
HJB-PDE	**H**amilton-**J**acobi-**B**ellman-**P**artial-**D**ifferential-**E**quation

Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	n-dimensional real vector space
$\succ$	Lexicographic ordering
$\|u\|_1$	L_1-norm of input u
$\|u\|_2^2$	Squared L_2-norm of input u
β	Input resource (fuel) budget
$\bar{\beta}$	Minimum fuel budget
$\boldsymbol{\beta}^*$	Total minimum fuel budget
$\mathcal{A}^\beta$	Attainable set under resource constraint β
$\mathcal{A}^\infty$	Attainable set without resource constraints
$\mathcal{R}^\beta$	Reachable set under resource constraint β
$\mathcal{X}_c^\beta$	Consensus region under resource constraint β
$\partial\mathcal{A}$	Boundary of the set $\mathcal{A}$
$\cap$	Set intersection
$\cup$	Set union
$\subseteq$	Subset relation
$\emptyset$	Empty set
$\mathbb{Q}$	Set of rational numbers
H	Hamiltonian
u^*	Optimal control input
$\bar{t}_f$	Consensus time
$\bar{\mathbf{x}}$	Consensus state
${}^N C_r$	Number of combinations of r elements from N