

RELIABILITY BASED ROBUST DESIGN OPTIMIZATION OF STRUCTURES

MOHD AMAN KHALID



**DEPARTMENT OF CIVIL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY DELHI**

MAY 2024

©Indian Institute of Technology Delhi (IITD), New Delhi, 2024
All rights reserved.

RELIABILITY BASED ROBUST DESIGN OPTIMIZATION OF STRUCTURES

by

MOHD AMAN KHALID

Department of Civil Engineering

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAY 2024

DEDICATION

This thesis is dedicated to my parents

Fareena and Khalid

for their endless love and unwavering support,

and to my siblings

Zoya and Rizwan

for their encouragement and empathy.



DEPARTMENT OF CIVIL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY DELHI
NEW DELHI-110016, DELHI, INDIA

Dr. Sahil Bansal

Professor

Email: sahil@iitd.ac.in

Phone: +91-11-2654-8414

CERTIFICATE

This is to certify that the thesis titled “**Reliability Based Robust Design Optimization of Structures,**” submitted by **Mohd Aman Khalid (Entry Number: 2019CEZ8234)**, to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy**, is a bonafide record of the research work done by him under my supervision in the Department of Civil Engineering, Indian Institute of Technology Delhi. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Date: 20/05/2024
Place: Delhi

Dr. Sahil Bansal

ACKNOWLEDGEMENT

I feel privileged and honored to sincerely thank my sincere thanks to my esteemed supervisor, *Prof. Sahil Bansal*. His consistent support and guidance have played a pivotal role in shaping this work. Prof. Bansal's impressive expertise, extensive knowledge, and commitment to academic excellence have served as a constant source of motivation throughout my research journey. The insightful feedback, constructive criticism, unwavering assistance, and encouragement he provided have propelled me to exceed my perceived limitations, allowing me to explore new dimensions in the field of research. Under his mentorship, I deepened my understanding of the subject at hand and refined essential skills that will undoubtedly benefit me in future endeavors. I am profoundly grateful for the opportunity to work under such an exceptional mentor.

I would also like to thank my research committee members *Prof. Ashok K. Keshari* (Chairman), *Prof. Alok Madan* (Department expert), and *Prof. Vikas Vikram Singh* (External expert) for encouraging and sharing insightful suggestions for improving the quality of my research work from various perspectives. I am also thankful to the reviewers of this thesis. I express my sincere thanks to the Head of the Department, *Prof. Vasant Matsagar*, and all the respected faculty members of the Department of Civil Engineering. I also appreciate the support given by the staff of the Department of Civil Engineering.

I would like to express my gratitude to my esteemed professors, *Dr. Mohammed Arif* and *Dr. Mohd Shariq* for their invaluable support and encouragement throughout my Ph.D. journey. I especially thank my colleagues *Dr. Jyotish Kumar Das*, *Eamon Karim Henikish*, *Raj Purohit Kiran*, and *Shakir Rather* for their support in a special way during my Ph.D. journey. I also thank *Dr. Jnyanendra Kumar Prusty*, *Dr. Kamal Anand*, *Dr. Mohammad Zakir*, and *Ms. Himasmita Das* for their valuable support. I would also like to

thank my friends *Zaki, Shahzar, Rahul, Mohsin, and Muneeb* for the fun time we spent together and for their support in my research work and personal life during this journey, which was unforgettable. I would also like to thank *Mr. Salman Jameel*, mess manager of my Karakoram hostel, for being with me; because of him, I never felt far from home during this long journey. I also thank all my lab mates from the institute for their support in various ways.

Most importantly, I am grateful to the almighty and extend my profound thanks to my family for their support, understanding, and love, without whom this thesis would not have been accomplished. I am truly thankful.

Date: 18.05.2024

Place: New Delhi



Mohd Aman Khalid

ABSTRACT

In most engineering applications, the traditional approaches for system designing entail taking into account deterministic system models and parameters. However, knowledge of the system parameters is never impeccable and thus rationally should be considered uncertain. The deterministic design approaches do not account for these uncertainties, such as those in loading, material, geometry, boundary conditions, etc. In general, uncertainties are measured by employing simplifying assumptions, such as taking into account extreme or average values and/or incorporating safety margins. Experience and engineering judgment are used to develop these hypotheses. While traditional techniques have proven effective in numerous practical design scenarios, the deterministic model often results in either excessive design precautions or failure when faced with extreme conditions. Therefore, an adequate level of reliability is essential when designing a system; other design goals may be important as well, such as there is an increasing demand for robust systems that are safer and less susceptible to uncertainties. The need to incorporate robustness and reliability into the design of systems is also recognized by the industry and commercial sectors. Therefore, developing the approaches for performing system design optimization while considering the uncertainties is the primary objective of this study.

Typical design methodologies that take uncertainties into account are broadly categorized as (a) Reliability-Based Design Optimization (RBDO) and (b) Robust Design Optimization (RDO). The RBDO method characterizes uncertain variables and failure modes in order to optimize designs for higher reliability. RDO, on the other hand, minimizes performance variance without eliminating the sources of variation. The primary challenges that prohibit the implementation of existing approaches for RDO and/or RBDO of structural systems are lower accuracy, the necessity of evaluating the multidimensional integrals involved in evaluating the robustness measure explicitly as a function of design parameters,

and limited applicability for problems with a higher number of uncertain variables. This is owing to the approximations and assumptions employed in computing the multidimensional integrals involved in evaluating the robustness measure and failure probability in RDO and RBDO, respectively. Furthermore, these design methodologies recurrently involve problems of multiple cycles of system evaluation within the optimization loops; the computational costs associated with them are very high, and in general, efficiency is favored to the detriment of accuracy.

During the course of this study, the aforementioned limitation has been successfully exploited, and broadly, four significant probabilistic stochastic simulation-based design approaches are developed, including (a) improving an existing stochastic simulation approach known as Stochastic Subset Optimization (SSO) that can be effectively utilized for the system design optimization both for deterministic and stochastic systems, (b) a simulation-based RDO approach that avoids the direct evaluation of the mean and variance of the performance function explicitly as a function of design parameters, (c) a simulation-based approach for efficiently handling the probabilistic constraint for Reliability-Based Design Optimization (RBDO), and lastly (d) a simulation-based approach for performing Reliability-Based Robust Design Optimization (RBRDO). The RBRDO aims to find the best design performance that guarantees the system robustness of optimal designs, simultaneously satisfying certain reliability requirements. Several high-dimensional realistic linear as well as nonlinear complex systems are investigated in the present study. The developed approaches are intended to be general (without any presumptions), with the goal of attaining a reasonable balance of accuracy and computational efficiency.

Keywords: augmented formulation, robust design, variance minimization, stochastic subset optimization, optimization under uncertainty, uncertain variables, Voronoi tessellation, stochastic simulation, stochastic optimization, robust

design optimization (RDO), reliability-based design optimization (RBDO), subset simulation, probabilistic constraint, decoupling, reliability-based robust design optimization (RBRDO).

अधिकांश इंजीनियरिंग अनुप्रयोगों में, सिस्टम डिजाइनिंग के लिए पारंपरिक दृष्टिकोण नियतात्मक सिस्टम मॉडल और मापदंडों को ध्यान में रखते हैं। हालाँकि, सिस्टम मापदंडों का ज्ञान कभी भी त्रुटिहीन नहीं होता है और इस प्रकार तर्कसंगत रूप से इसे अनिश्चित माना जाना चाहिए। नियतात्मक डिज़ाइन दृष्टिकोण इन अनिश्चितताओं को ध्यान में नहीं रखते हैं, जैसे कि लोडिंग, सामग्री, ज्यामिति, सीमा की स्थिति आदि। सामान्य तौर पर, अनिश्चितताओं को सरलीकृत धारणाओं को नियोजित करके मापा जाता है, जैसे कि चरम या औसत मूल्यों को ध्यान में रखना और/या सुरक्षा मार्जिन को शामिल करना। इन परिकल्पनाओं को विकसित करने के लिए अनुभव और इंजीनियरिंग निर्णय का उपयोग किया जाता है। जबकि पारंपरिक तकनीकें कई व्यावहारिक डिज़ाइन परिदृश्यों में प्रभावी साबित हुई हैं, नियतात्मक मॉडल अक्सर चरम स्थितियों का सामना करने पर या तो अत्यधिक डिज़ाइन सावधानियों या विफलता का परिणाम होता है। इसलिए, किसी सिस्टम को डिज़ाइन करते समय विश्वसनीयता का पर्याप्त स्तर आवश्यक है; अन्य डिज़ाइन लक्ष्य भी महत्वपूर्ण हो सकते हैं, जैसे कि मजबूत प्रणालियों की मांग बढ़ रही है जो सुरक्षित हैं और अनिश्चितताओं के प्रति कम संवेदनशील हैं। सिस्टम के डिज़ाइन में मजबूती और विश्वसनीयता को शामिल करने की आवश्यकता को उद्योग और वाणिज्यिक क्षेत्रों द्वारा भी मान्यता दी गई है। इसलिए, अनिश्चितताओं पर विचार करते हुए सिस्टम डिज़ाइन अनुकूलन करने के लिए दृष्टिकोण विकसित करना इस अध्ययन का प्राथमिक उद्देश्य है।

विशिष्ट डिज़ाइन पद्धतियाँ जो अनिश्चितताओं को ध्यान में रखती हैं, उन्हें मोटे तौर पर (ए) विश्वसनीयता-आधारित डिज़ाइन ऑप्टिमाइज़ेशन (आरबीडीओ) और (बी) मजबूत डिज़ाइन ऑप्टिमाइज़ेशन (आरडीओ) के रूप में वर्गीकृत किया गया है। उच्च विश्वसनीयता के लिए डिज़ाइन को ऑप्टिमाइज़ेशन करने के लिए आरबीडीओ विधि अनिश्चित चर और विफलता मोड की विशेषता बताती है। दूसरी ओर, आरडीओ भिन्नता के स्रोतों को समाप्त किए बिना प्रदर्शन भिन्नता को कम

करता है। प्राथमिक चुनौतियाँ जो संरचनात्मक प्रणालियों के आरडीओ और/या आरबीडीओ के लिए मौजूदा दृष्टिकोणों के कार्यान्वयन पर रोक लगाती हैं, वे हैं कम सटीकता, डिज़ाइन मापदंडों के एक कार्य के रूप में स्पष्ट रूप से मजबूती के माप का मूल्यांकन करने में शामिल बहुआयामी अभिन्नो का मूल्यांकन करने की आवश्यकता, समस्याओं के लिए सीमित प्रयोज्यता, और अनिश्चित चरों की अधिक संख्या। यह क्रमशः आरडीओ और आरबीडीओ में मजबूती माप और विफलता संभावना के मूल्यांकन में शामिल बहुआयामी अभिन्नो की गणना में नियोजित अनुमानों और मान्यताओं के कारण है। इसके अलावा, इन डिज़ाइन पद्धतियों में बार-बार ऑप्टिमाइज़ेशन लूप के भीतर सिस्टम मूल्यांकन के कई चक्रों की समस्याएं शामिल होती हैं; उनसे जुड़ी कम्प्यूटेशनल लागत बहुत अधिक है, और सामान्य तौर पर, सटीकता की हानि के लिए दक्षता को प्राथमिकता दी जाती है।

इस अध्ययन के दौरान, उपर्युक्त सीमा का सफलतापूर्वक उपयोग किया गया है, और मोटे तौर पर, चार महत्वपूर्ण संभाव्य स्टोकेस्टिक सिमुलेशन-आधारित डिज़ाइन दृष्टिकोण विकसित किए गए हैं, जिनमें (ए) स्टोकेस्टिक सबसेट ऑप्टिमाइज़ेशन (एसएसओ) के रूप में ज्ञात मौजूदा स्टोकेस्टिक सिमुलेशन दृष्टिकोण में सुधार करना शामिल है जो कर सकता है नियतात्मक और स्टोकेस्टिक सिस्टम दोनों के लिए सिस्टम डिज़ाइन अनुकूलन के लिए प्रभावी ढंग से उपयोग किया जाना चाहिए, (बी) एक सिमुलेशन-आधारित आरडीओ दृष्टिकोण जो डिज़ाइन पैरामीटर के फ़ंक्शन के रूप में स्पष्ट रूप से प्रदर्शन फ़ंक्शन के माध्य और भिन्नता के प्रत्यक्ष मूल्यांकन से बचाता है, (सी) ए विश्वसनीयता-आधारित डिज़ाइन ऑप्टिमाइज़ेशन (आरबीडीओ) के लिए संभाव्य बाधा को कुशलतापूर्वक संभालने के लिए सिमुलेशन-आधारित दृष्टिकोण, और अंत में (डी) विश्वसनीयता-आधारित मजबूत डिज़ाइन ऑप्टिमाइज़ेशन (आरबीआरडीओ) करने के लिए सिमुलेशन-आधारित दृष्टिकोण। आरबीआरडीओ का लक्ष्य सर्वोत्तम डिज़ाइन प्रदर्शन ढूँढना है जो इष्टतम डिज़ाइन की सिस्टम मजबूती की गारंटी देता है, साथ ही कुछ विश्वसनीयता आवश्यकताओं को भी पूरा करता है। वर्तमान अध्ययन में कई उच्च-आयामी यथार्थवादी रैखिक और गैर-रेखीय जटिल प्रणालियों की जांच की गई है। विकसित दृष्टिकोण

सटीकता और कम्प्यूटेशनल दक्षता का उचित संतुलन प्राप्त करने के लक्ष्य के साथ सामान्य (बिना किसी अनुमान के) होने का इरादा रखते हैं।

TABLE OF CONTENTS

Content	Page No.
CERTIFICATE	i
ACKNOWLEDGEMENT	ii
ABSTRACT	iv
TABLE OF CONTENTS	x
LIST OF FIGURES	xiii
LIST OF TABLES	xv
LIST OF ABBREVIATIONS AND SYMBOLS	xvii
Chapter 1	1
1. Introduction	1
1.1. Background and Motivation	1
1.2. Challenges in Optimization under Uncertainty	6
1.3. Thesis Objectives and Organization	9
Chapter 2	12
2. Improved Stochastic Subset Optimization Method for Structural Design Optimization	12
2.1. Background and Motivation	12
2.2. Proposed Approach	16
2.2.1. Partitioning of design space	17
2.2.2. Identification of an optimal subset	18
2.2.3. Simulation of conditional samples	19
2.2.4. Stopping criteria	20
2.2.5. Implementation Issues	21
2.3. Special case: Deterministic optimization	22
2.4. Illustrative Examples	23
2.4.1. Multimodal deterministic optimization problems	24
2.4.2. Robust design optimization of the Tuned Mass Damper	29
2.4.3. 120-bars truss structure	32
2.4.4. Reliability-based design of a base isolated structure	33
2.5. Summary	35

Chapter 3	37
3. An Augmented Formulation for Robust Design Optimization of Structures	37
3.1. Background and Motivation	37
3.2. Problem Formulation	41
3.3. Augmented Formulation for Mean Minimization	43
3.4. Augmented Formulation for Variance Minimization	44
3.5. Design Space for Additional Uncertain Variable	47
3.6. Stopping Criteria	49
3.7. Implementation Issues	50
3.8. Direct Search Optimization Via Ranking and Selection	50
3.9. Illustrative Examples	54
3.9.1. Four-bar linear truss structure	55
3.9.2. 25-bar truss structure with material nonlinearity	58
3.9.3. 120-bar truss structure with geometric nonlinearity	62
3.9.4. TMD attached to a SDOF primary structure	64
3.10. Computational Cost Comparison	69
3.11. Summary	71
Chapter 4	72
4. An Approach for Reliability-Based Design Optimization of Stochastic Systems	72
4.1. Background and Motivation	72
4.2. Augmented Formulation for Reliability Constraint	77
4.3. Proposed Approach	79
4.4. RBDO Problem with Multiple Reliability Constraints	82
4.5. Number of samples N_i	87
4.6. Implementation Issues	87
4.7. Deterministic Optimization to Minimize Cost Function	88
4.8. Illustrative Examples	89
4.8.1. Mathematical functions	89
4.8.1.1. Function – 1: Two-dimensional problem with single reliability constraint	89

4.8.1.2. Function - 2: Two-dimensional problem with multiple reliability constraint	93
4.8.1.3. Function - 3: Ten-dimensional problem with multiple reliability constraint	96
4.8.2. Straight column design	97
4.8.3. Ten-bar truss structure design problem	99
4.8.4. Damper design for a four-story structure system	101
4.9. Summary	104
Chapter 5	106
5. A Stochastic Simulation Framework for Reliability-Based Robust Design Optimization of Stochastic Systems	106
5.1. Background and Motivation	106
5.2. Proposed Approach	112
5.3. Illustrative Examples	113
5.3.1. Four bars truss structure	114
5.3.2. Ten bars truss structure with multiple reliability constraints	117
5.4. Summary	119
Chapter 6	121
6. Research Contribution and Future Scope	121
6.1. Assumptions and Limitations	124
6.2. Recommendations for Future Work	126
References	127
Appendices	163
Appendix – A: Voronoi Tessellation	163
Appendix – B: Stochastic Subset Optimization	165
Appendix – C: Simulation of Samples Following Linear Constraint	169
Appendix – D: Sampling Techniques	171
Appendix – E: Sample Average Approximation	173
Publications from the Thesis Work	174
Course work	175
Curriculum Vitae	176

LIST OF FIGURES

Fig. No.	Figure Caption	Page. No.
Fig. 1:	Concept of RDO.....	5
Fig. 2:	Illustration of the proposed iSSO algorithm.	20
Fig. 3:	Implementation of Voronoi tessellation in iSSO.	22
Fig. 4:	Results for the Griewank function.	25
Fig. 5:	Results for the Cross-in-Tray function.....	26
Fig. 6:	Results for the Holder Table function.....	27
Fig. 7:	Comparison of double sort algorithm and Genetic algorithm results.	29
Fig. 8:	Percentage change in volume at each iSSO iteration.....	29
Fig. 9:	TMD attached to an SDOF system.	30
Fig. 10:	Illustration of the identified admissible design at various iterations of SSO.....	31
Fig. 11:	120-bar dome truss structure.....	32
Fig. 12:	(left) 10-story base isolated shear model, and (right) force-deformation of bilinear isolator [166].	33
Fig. 13:	Four-bar truss structure.	56
Fig. 14:	(a) Response mean μ_u versus A_1 , (b) histogram for A_1 samples distributed according to $J(A_1)$	56
Fig. 15:	(a) Response variance of σ_u^2 versus A_1 , (b) histogram for $[A_1, \bar{\varphi}]$ samples distributed as $J(A_1, \bar{\varphi})$	57
Fig. 16:	The Pareto front (example 1).	57
Fig. 17:	Four bars truss structure results for the variance minimization.	58
Fig. 18:	25-bar truss structure.....	59
Fig. 19:	Volume of the set $\hat{I}_k$ versus iSSO generation number.	60
Fig. 20:	120-bar dome truss structure.....	62
Fig. 21:	The primary SDOF structure with the TMD.....	65
Fig. 22:	Mean minimization results for different mass ratios and c.o.v values.....	66
Fig. 23:	Variance minimization results for different mass ratios and c.o.v values.	67
Fig. 24:	Optimal TMD design parameters for mean and variance minimization ($\alpha = 0.5$) at different mass ratios and c.o.v values	68

Fig. 25: Objective function $\mu_{R_{zz}}$ and $\sigma_{R_{zz}}^2$ values for mean and variance minimization ($\alpha = 0.5$) at different mass ratios and c.o.v values.	69
Fig. 26: Demonstration of the proposed approach for identifying design subspace.	81
Fig. 27: Demonstration of the proposed approach for identifying design subspace in RBDO problem with multiple reliability constraints at iteration – 1.....	85
Fig. 28: Demonstration of the proposed approach for identifying design subspace in RBDO problem with multiple reliability constraints at iteration – 2.....	86
Fig. 29: Results of mathematical function – 1.....	90
Fig. 30: Failure probability for different design configurations, $-\log_{10} P_F(\boldsymbol{\varphi})$, at iteration – 1.....	91
Fig. 31: Failure probability for different design configurations, $-\log_{10} P_F(\boldsymbol{\varphi})$, at iteration – 2.....	92
Fig. 32: Failure probability for different design configurations, $-\log_{10} P_F(\boldsymbol{\varphi})$, for mathematical function – 2.....	94
Fig. 33: Straight column example subjected to axial stochastic load.....	98
Fig. 34: 10-bars truss structure.	100
Fig. 35: Schematic diagram of a Linear Viscous Dampers - MDOF structure.	103
Fig. 36: Identification of feasible design space satisfying reliability constraints in the first stage of RBRDO.	113
Fig. 37: RDO within the identified feasible design space in the second stage of RBRDO.	113
Fig. 38: Four-bar truss structure.	114
Fig. 39: Illustration of the four bars truss structure example.	116
Fig. 40: 10-bar truss structure.....	117
Fig. 41: Voronoi diagram in 2-dimensional space.	163
Fig. 42: Illustration of the SSO algorithm.	168
Fig. 43: Sampling a random point in a triangle.	170

LIST OF TABLES

Table No.	Table Caption	Page. No.
Table 1:	Statistics of optimization results for multimodal deterministic optimization problems.....	28
Table 2:	Variance minimization of TMD-structure.	31
Table 3:	Results for the 120 bars truss structure.	32
Table 4:	Base isolation structure system optimization results (best of 10 independent runs)	35
Table 5:	Mean minimization ($\alpha = 1$) results, best of 10 runs (25 bar truss).....	60
Table 6:	Variance minimization ($\alpha = 0$) results, best of 10 runs (25 bar truss).....	61
Table 7:	Mean and Variance minimization ($\alpha = 0.5$) results, best of 10 runs (25 bar truss)	61
Table 8:	Mean minimization ($\alpha = 1$) results, best of 10 runs (120 bar truss).....	63
Table 9:	Variance minimization ($\alpha = 0$) results, best of 10 runs (120 bar truss).....	63
Table 10:	Mean and Variance minimization ($\alpha = 0.5$) results, best of 10 runs (120 bar truss).....	64
Table 11:	Computational demand comparison of different approaches.	70
Table 12:	Results for the mathematical function – 1.	92
Table 13:	Statistics of mathematical function - 1 optimization results (50 independent runs).	93
Table 14:	Results for the mathematical function – 2.	95
Table 15:	Statistics of mathematical function - 2 optimization results (50 independent runs).	95
Table 16:	Results for the mathematical function – 3.	97
Table 17:	Statistics of mathematical function - 3 optimization results (50 independent runs)	97
Table 18:	Results for the straight column structure.	99
Table 19:	Statistics of straight column structure optimization results (50 independent runs)	99
Table 20:	10-bars truss structure optimization results	101
Table 21:	Statistics of 10-bar truss structure optimization results (50 independent runs).	101

Table 22: Linear viscous damper - MDOF structure optimization results	104
Table 23: Statistics of Linear viscous dampers - MDOF structure optimization results (50 independent runs).....	104
Table 24: Results for the four bars truss structure.	116
Table 25: Statistics of four bars truss structure optimization results (50 independent runs)	116
Table 26: Comparison of results obtained from the different stochastic design methodologies.....	117
Table 27: 10-bars truss structure optimization results for RBRDO.....	118
Table 28: Statistics of Linear viscous dampers - MDOF structure optimization results (50 independent runs).....	119

LIST OF ABBREVIATIONS AND SYMBOLS

UQ	Uncertainty Quantification
RBDO	Reliability-Based Design Optimization
RDO	Robust Design Optimization
RBRDO	Reliability-Based Robust Design Optimization
MCS	Monte-Carlo Simulation
LSF	Limit State Functions
SSO	Stochastic Subset Optimization
SAA	Sample Average Approximation
TSE	Taylor Series Expansion
PDF	Probability Density Function
SS	Subset Simulation
TMD	Tuned Mass Dampers
iSSO	Improved Stochastic Subset Optimization
GA	Genetic Algorithm
PSO	Particle Swarm Optimization
GBA	Gradient-Based Optimization Approach
NF	Number of Failure
NS	Number of Success
BV	Best Value
WV	Worst Value
AV	Average Value
c.o.v	Coefficient of Variation
FE	Number of Function Evaluations

Gen	Generations
VR	Volume Reduction Percentage
SDOF	Single Degree of Freedom
RBF	Radial Basis Function
S/N	Signal to Noise
FPFs	Failure Probability Functions
RIA	Reliability Index Approach
PMA	Performance Measure Approach
SLSV	Single-Loop Single-Vector
SORA	Sequential Optimization and Reliability Analysis
NP-SSO	Non-Parametric Stochastic Subset Optimization
KDE	Kernel Density Estimation
ADA	Adaptive Decoupling Method
DRM	Dimension Reduction Method
$\boldsymbol{\varphi}$	Design parameters vector
$\boldsymbol{\theta}$	Uncertain variables vector
Φ	Parameter space of design parameters
Θ	Parameter space of uncertain variables
$\mathbb{R}$	Set of real numbers
n_{φ}	Number of design parameters
n_{θ}	Number of uncertain variables
$[\cdot]^T$	Vector transpose
$g(\cdot)$	Performance function
$P_F(\cdot)$	Probability of failure

$\mu_g(.)$	Mean of the performance function $g(.)$
$\sigma_g^2(.)$	Variance of the performance function $g(.)$
$h(.)$	Cost function
$E[.]$	Expectation
$\pi(.)$	Auxiliary PDF
ε	Tolerance limit
$I(.)$	Indicator function