

THEORETICAL ANALYSIS OF HAWKES AND DYNAMIC CONTAGION PROCESSES WITH THEIR APPLICATIONS

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Theoretical Analysis of Hawkes and Dynamic Contagion Processes with their Applications

by

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Submitted

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Doctor of Philosophy

to the



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*Dedicated to
my grandfather*

Certificate

This is to certify that the thesis entitled **Theoretical Analysis of Hawkes and Dynamic Contagion Processes with their Applications** submitted by **Ms. SHAMIKSHA** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results presented in this thesis have not been submitted, in whole or in part, to any other university or institution for the award of any degree or diploma.

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Abstract

This thesis investigates the theoretical aspects of Hawkes and dynamic contagion processes, and their generalizations. These processes constitute a class of self-exciting point processes that model events in which the occurrence of one event increases the likelihood of future events. This characteristic makes them particularly effective for capturing temporally dependent and clustered event data. This is the reason why, they have wide-ranging applications across various fields, including finance (where trades or shocks trigger subsequent activity), seismology (where earthquakes trigger aftershocks), neuroscience (where neuronal spikes trigger subsequent firings), and social networks (where posts or messages encourage further interactions). Motivated by this broad applicability, this thesis offers a fundamental analysis of essential properties, such as stability, stationarity, the Markov property, and long-run behavior, along with their applications. The rigorous theoretical results presented enhance the significance of Hawkes and dynamic contagion processes in modeling complex temporal phenomena.

In the theoretical analysis of Hawkes processes, stability condition is the necessary tool to ensure that the process does not explode with infinitely many events in a finite time. While the Stationarity condition guarantees that the statistical properties remain consistent over time. As discussed, the main property of a Hawkes process is its self-exciting property; therefore, the Markov property doesn't hold in general. However, when holds, it ensures analytical tractability and simplifies both theoretical analysis and computational methods. Moreover, asymptotic results such as the law of large numbers, the central limit theorem, and the large deviation principle play a crucial role in understanding their long-run behavior. These results simplify the average dynamics, stochastic fluctuations, and probabilities of rare events in the processes, and they are important for understanding both typical and extreme event patterns.

The linear Hawkes process admits a cluster (or immigration–birth) representation. The work in this thesis extends the classical Hawkes model by incorporating more general mechanisms through which

past events can influence future ones. This is achieved by examining alternative forms of event interactions within an immigration–birth framework. Such theoretical exploration is crucial, as it not only enhances the mathematical understanding of the process but also increases their adaptability and relevance in applications where event dynamics are more complex than those addressed by standard models.

Moreover, this thesis examines the interaction between the Cox–Ingersoll–Ross (CIR) process and the dynamic contagion process. In this study, the dynamics of the CIR process are augmented by the jump components that are influenced by contagion effects. The proposed framework effectively captures the mean-reverting characteristics of the CIR process as well as the possibility of sudden increases in intensity triggered by past events. This phenomenon is particularly significant in credit risk scenarios, where defaults among connected firms may occur in clusters. Thus, to better understand the implications of this behavior, this thesis analysis the asymptotic results of the proposed model.

When the Hawkes process models the occurrence of self-exciting events only over time, without considering spatial dimensions, it is referred to as a temporal Hawkes process. However, there are many real-world applications that involve the study of the spatio-temporal structure of Hawkes processes, such as earthquake aftershocks, crime events, disease spread, and information diffusion. Investigating these applications provides a framework for understanding how localized events can trigger cascades that propagate both spatially and temporally, resulting in complex clustering patterns. In particular, this thesis investigates the theoretical properties of such processes, where spatial and temporal dynamics are inseparably linked.

Furthermore, infinite-server models are widely used to represent large-scale service systems in which many events can occur and be processed simultaneously. Their significance increases when the arrivals are self-exciting or exhibit dynamic contagion dynamics. These characteristics capture the clustering and dependence patterns often observed in telecommunications, healthcare, and risk management. This thesis extends infinite-server models to incorporate such complex arrival behaviors and provides the more realistic and rigorous analytical framework. In addition, *Hawkes/M/C* self-exciting retrial queueing systems are examined to illustrate that, how feedback-driven service dynamics can be effectively modeled. All these contributions are thoroughly discussed in this thesis.

This thesis contributes to the existing literature by advancing the theoretical understanding of Hawkes and dynamic contagion processes. After presenting the fundamental properties of these processes, the thesis explores their essential extensions and highlights their significance in practical applications. In this way, this thesis improves both the mathematical insights and the practical usefulness of self-exciting processes for modeling complex real-world phenomena.

सारांश

यह शोध प्रबंध हॉक्स और गतिशील संक्रमण प्रक्रियाओं के सैद्धांतिक पहलुओं और उनके सामान्यीकरण की जांच करता है। ये प्रक्रियाएँ आत्म-उत्तेजक बिंदु प्रक्रियाओं के एक वर्ग का गठन करती हैं जो उन घटनाओं का मॉडल बनाती हैं जिनमें एक घटना की घटना भविष्य की घटनाओं की संभावना को बढ़ाती है। यह विशेषता उन्हें अस्थायी रूप से निर्भर और क्लस्टर किए गए घटना डेटा को पकड़ने के लिए विशेष रूप से प्रभावी बनाती है। यही कारण है कि उनके पास विभिन्न क्षेत्रों में व्यापक अनुप्रयोग हैं, जिसमें वित्त (जहां व्यापार या झटके बाद की गतिविधि को ट्रिगर करते हैं) भूकंप विज्ञान (जहां भूकंप आफ्टरशॉक को ट्रिगर करते हैं) तंत्रिका विज्ञान (जहां न्यूरोनल स्पाइक्स बाद में फायरिंग को ट्रिगर करते हैं) और सामाजिक नेटवर्क (जहां पोस्ट या संदेश आगे की बातचीत को प्रोत्साहित करते हैं) शामिल हैं। इस व्यापक प्रयोज्यता से प्रेरित, यह थीसिस उनके अनुप्रयोगों के साथ-साथ स्थिरता, स्थिरता, मार्कोव गुण और दीर्घकालिक व्यवहार जैसे आवश्यक गुणों का एक मौलिक विश्लेषण प्रदान करता है। प्रस्तुत किए गए कठोर सैद्धांतिक परिणाम जटिल लौकिक घटनाओं के प्रतिरूपण में हॉक्स और गतिशील संक्रमण प्रक्रियाओं के महत्व को बढ़ाते हैं।

हॉक्स प्रक्रियाओं के सैद्धांतिक विश्लेषण में, स्थिरता की स्थिति यह सुनिश्चित करने के लिए आवश्यक उपकरण है कि प्रक्रिया एक सीमित समय में असीम रूप से कई घटनाओं के साथ विस्फोट न करे। जबकि स्थिर स्थिति इस बात की गारंटी देती है कि सांख्यिकीय गुण समय के साथ सुसंगत रहते हैं। जैसा कि चर्चा की गई है, हॉक्स प्रक्रिया का मुख्य गुण इसकी आत्म-उत्तेजक गुण है; इसलिए, मार्कोव गुण सामान्य रूप से नहीं है। हालांकि, जब यह होता है, तो यह विश्लेषणात्मक ट्रैक्टैबिलिटी सुनिश्चित करता है और सैद्धांतिक विश्लेषण और कम्प्यूटेशनल विधियों दोनों को सरल बनाता है। इसके अलावा, बड़ी संख्या के नियम, केंद्रीय सीमा प्रमेय और बड़े विचलन सिद्धांत जैसे असिम्प्टोटिक परिणाम उनके दीर्घकालिक व्यवहार को समझने में महत्वपूर्ण भूमिका निभाते हैं। ये परिणाम औसत गतिशीलता, यादृच्छिक उतार-चढ़ाव और प्रक्रियाओं में दुर्लभ घटनाओं की संभावनाओं को सरल बनाते हैं, और वे विशिष्ट और चरम घटना पैटर्न दोनों को समझने के लिए महत्वपूर्ण हैं।

रैखिक हॉक्स प्रक्रिया एक समूह (या आप्रवासन-जन्म) प्रतिनिधित्व को स्वीकार करती है। इस थीसिस में काम अधिक सामान्य तंत्र को शामिल करके शास्त्रीय हॉक्स मॉडल का विस्तार करता है जिसके माध्यम से पिछली घटनाएँ भविष्य को प्रभावित कर सकती हैं। यह एक आप्रवासन-जन्म ढांचे के भीतर घटना बातचीत के वैकल्पिक

रूपों की जांच करके प्राप्त किया जाता है। इस तरह का सैद्धांतिक अन्वेषण महत्वपूर्ण है, क्योंकि यह न केवल प्रक्रिया की गणितीय समझ को बढ़ाता है, बल्कि उन अनुप्रयोगों में उनकी अनुकूलनशीलता और प्रासंगिकता को भी बढ़ाता है जहां मानक मॉडल द्वारा संबोधित घटनाओं की गतिशीलता अधिक जटिल होती है।

इसके अलावा, यह थीसिस Cox-Ingersoll-Ross (CIR) प्रक्रिया और गतिशील संक्रमण प्रक्रिया के बीच बातचीत की जांच करता है। इस अध्ययन में, CIR प्रक्रिया की गतिशीलता को कूद घटकों द्वारा संवर्धित किया जाता है जो संक्रामक प्रभावों से प्रभावित होते हैं। प्रस्तावित ढांचा प्रभावी रूप से CIR प्रक्रिया की मीन-रिवर्टिंग विशेषताओं के साथ-साथ पिछली घटनाओं से उत्पन्न तीव्रता में अचानक वृद्धि की संभावना को दर्शाता है। यह घटना विशेष रूप से ऋण जोखिम परिदृश्यों में महत्वपूर्ण है, जहां संबद्ध फर्मों के बीच चूक समूहों में हो सकती है। इस प्रकार, इस व्यवहार के निहितार्थ को बेहतर ढंग से समझने के लिए, यह थीसिस प्रस्तावित मॉडल के एसिम्प्टोटिक परिणामों का विश्लेषण करता है।

जब हॉक्स प्रक्रिया स्थानिक आयामों पर विचार किए बिना केवल समय के साथ आत्म-उत्तेजक घटनाओं की घटना का मॉडल बनाती है, तो इसे अस्थायी हॉक्स प्रक्रिया के रूप में संदर्भित किया जाता है। हालांकि, कई वास्तविक दुनिया के अनुप्रयोग हैं जिनमें हॉक्स प्रक्रियाओं की स्थानिक-अस्थायी संरचना का अध्ययन शामिल है, जैसे कि भूकंप के बाद के झटके, अपराध की घटनाएं, रोग का प्रसार और सूचना प्रसार। इन अनुप्रयोगों की जांच करना यह समझने के लिए एक ढांचा प्रदान करता है कि कैसे स्थानीयकृत घटनाएं कैस्केड को ट्रिगर कर सकती हैं जो स्थानिक और अस्थायी दोनों रूप से फैलती हैं, जिसके परिणामस्वरूप जटिल क्लस्टरिंग पैटर्न होते हैं। विशेष रूप से, यह थीसिस ऐसी प्रक्रियाओं के सैद्धांतिक गुणों की जांच करता है, जहां स्थानिक और लौकिक गतिकी अविभाज्य रूप से जुड़े हुए हैं।

इसके अलावा, अनंत-सर्वर मॉडल का व्यापक रूप से बड़े पैमाने पर सेवा प्रणालियों का प्रतिनिधित्व करने के लिए उपयोग किया जाता है जिसमें कई घटनाएं हो सकती हैं और एक साथ संसाधित की जा सकती हैं। उनका महत्व तब बढ़ जाता है जब आगमन आत्म-उत्तेजक होते हैं या गतिशील संक्रामक गतिशीलता प्रदर्शित करते हैं। ये विशेषताएं अक्सर दूरसंचार, स्वास्थ्य सेवा और जोखिम प्रबंधन में देखे जाने वाले क्लस्टरिंग और निर्भरता पैटर्न को पकड़ती हैं। यह शोध प्रबंध इस तरह के जटिल आगमन व्यवहारों को शामिल करने के लिए अनंत-सर्वर मॉडल का विस्तार करता है और अधिक यथार्थवादी और कठोर विश्लेषणात्मक ढांचा प्रदान करता है। इसके अलावा, Hawkes/M/C स्व-उत्तेजक पुनः परीक्षण कतार प्रणालियों की जांच की जाती है ताकि यह स्पष्ट किया जा सके कि प्रतिक्रिया-संचालित सेवा गतिशीलता को प्रभावी ढंग से कैसे मॉडल किया जा सकता है। इन सभी योगदानों पर इस थीसिस में विस्तार से चर्चा की गई है।

यह शोध प्रबंध हॉक्स और गतिशील संक्रमण प्रक्रियाओं की सैद्धांतिक समझ को आगे बढ़ाकर मौजूदा साहित्य में योगदान देता है। इन प्रक्रियाओं के मौलिक गुणों को प्रस्तुत करने के बाद, थीसिस उनके आवश्यक विस्तार की खोज करता है और व्यावहारिक अनुप्रयोगों में उनके महत्व पर प्रकाश डालता है। इस तरह, यह थीसिस जटिल वास्तविक दुनिया की घटनाओं के मॉडलिंग के लिए गणितीय अंतर्दृष्टि और आत्म-उत्तेजक प्रक्रियाओं की व्यावहारिक उपयोगिता दोनों में सुधार करता है।

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