Unit Group of Rings and Group Rings

By

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Group theory; Rings and modules.

To My Parents

CERTIFICATE

I am satisfied that the thesis entitled *Unit Group of Rings and Group Rings* presented by Ms. Pooja Yadav (2005MAZ8115) is worthy of consideration for the award of the degree of Doctor of philosophy and is a record of the bonafide research work carried out by her under my guidance and supervision and the results contained in it have not been submitted in part or full to any other university or Institute for award of any degree/diploma.

R. K. Sharma

Prof. R. K. Sharma

(Supervisor)

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Abstract

The aim of the thesis is to study the unit group of rings and group rings. We focus mainly on the generators and presentations of linear groups. An element a of a ring R is said to be unit regular if $a = aua$ for some unit u in R. Equivalently, a is unit regular if and only if $a = eu$ for some idempotent e and some unit u in R. An element $a \in R$ is called clean if $a = e + u$ for some idempotent e and some unit u in R. Here, we introduce the concept of Lie regular elements. An element a of a ring R is said to be a **Lie regular** if $a = [e, u] = eu - ue$, where e is an idempotent in R and u is a unit of R. Further, a unit in R is said to be **Lie regular unit** if it is Lie regular as an element of R. In this thesis, we study Lie regular units and show the existence of Lie regular elements and Lie regular units in 2×2 matrix rings. Also, we have shown that any Lie regular element in $M_2(R)$, where R is a commutative domain in which 2 is invertible and in the group algebra KD_∞ of infinite dihedral group over a field of characteristic not equal to 2, can be expressed as $[u_1, u_2]$, where u_1 and u_2 are units. We obtain Lie regular units as generators of linear groups $GL(2, \mathbb{Z}_p)$, $GL(2, \mathbb{Z}_{2^n})$, $GL(2, \mathbb{Z}_{p^n})$, $GL(2, \mathbb{Z}_{2p^n})$, $GL(2, \mathbb{Z}_{3p^n})$ and $GL(2, \mathbb{Z}_{5p^n})$, where p is any odd prime. Presentations of linear groups $GL(2, \mathbb{Z}_4)$, $GL(2, \mathbb{Z}_6)$, $GL(2, \mathbb{Z}_8)$, $GL(2, \mathbb{Z}_9)$, $GL(2, \mathbb{Z}_{10})$, $GL(2, \mathbb{Z}_{12})$, $GL(2, \mathbb{Z}_{14})$, $GL(2, \mathbb{Z}_{15})$, $GL(2, \mathbb{Z}_{22})$, $GL(2, \mathbb{Z}_{25})$, $GL(2, \mathbb{Z}_{26})$, $GL(2, \mathbb{Z}_{27})$ and $GL(2, \mathbb{Z}_{34})$ are also given in terms of Lie regular units as generators.

We have studied circulant matrices, their properties and equivalent results to obtain the structure of the unit group of integral group rings. We have given a complete characterization of the unit groups $\mathcal{U}(Cr_3(F_{p^k}))$, $\mathcal{U}(Cr_4(F_{p^k}))$ and $\mathcal{U}(Cr_5(\mathbb{Z}_p))$.

Further, we have obtained presentations of the unit groups $\mathcal{U}(\mathbb{Z}_2Q_8)$ and $\mathcal{U}(\mathbb{Z}_2D_8)$. We have described a complete characterization of the unit group $\mathcal{U}(\mathbb{Z}_pQ_8)$. We have given a matrix representation for the group algebra $R(C_2 \times D_\infty)$. In particular, we have shown that $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ is not finitely generated.

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