UNITS OF THE GROUP RINGS OF DIRECT AND SEMI-DIRECT PRODUCTS

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Units of the Group Rings of Direct and Semi-direct Products

by

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Dedicated to
My Love, My Family!

Thesis Certificate

This is to certify that the thesis entitled **Units of the Group Rings of Direct and Semi-direct Products**, submitted by **Abhinay Kumar Gupta (2018MAZ8258)**, to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy**, is a bonafide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Abstract

KEYWORDS: Group Ring; Integral Group Ring; Unit Group; Trivial Units; Finite Fields; Polynomials; General Linear Group.

Let G be a group and ${C_2}^n$ denotes of direct product of n copies of the cyclic group C_2 with two elements. We introduce a characterization of the unit group of the integral group ring $U(\mathbb{Z}[G \times {C_2}^n])$ in terms of the unit group $U(\mathbb{Z}G)$ for every positive integer n. Specifically, we proved that $U(\mathbb{Z}[G \times {C_2}^n]) \cong M_{x_n} \rtimes (\cdots \rtimes (M_{x_2} \rtimes (M_{x_1} \rtimes U(\mathbb{Z}G))))$ for an arbitrary group G. Moreover, we explicitly describe M_{x_n} for each positive integer n as $M_{x_n} = \{1 + (x_n - 1)(\sum_{i_1,i_2,\cdots,i_{n-1}} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_{n-1}} R_{i_1 i_2 \cdots i_{n-1}}) : A_n \in GL(2^k,\ \mathbb{Z}G),\ R_{i_1 i_2 \cdots i_{n-1}} \in \mathbb{Z}G$ for each $i_1, i_2, \cdots, i_{n-1} \in \{0,\ 1\}\}$. Hereby to every M_{x_n} a matrix A_n is associated. For each $n = 2, 3, \cdots$, we determine some conditions, for which the class of matrices of the type A_n to belong in $GL(2^{n-1}, \mathbb{Z}G)$. Furthermore, for every matrix A_n, a new kind of matrix T_n is defined by replacing the entries of matrix A_n by commuting indeterminates in some order, after that a specific symmetric polynomial f_n with integer coefficients is being associated with each matrix T_n as $f_n(X_1, X_2, \cdots, X_{2^{n-1}}) = \frac{1}{2^n}(\det(T_n) - 1)$. And of these, the integer solution is described.

We also characterize the unit group $U(\mathbb{Z}G)$ for the meta-cyclic group of the form $G = \langle x, c : x^3 = 1, c^n = 1, cxc^{-1} = x^{-1} \rangle$. Consequently, we derived an interesting outcome by establishing a one-to-one correspondence between elements in the set of integer solutions of some explicit polynomials and elements in a subgroup of the finite index within the corresponding normalized unit group for the symmetric group S_3 and the dicyclic group Dic_3. We also explicitly describe each unit of the group ring $Z_p S_3$ for each positive prime $p \geq 5$, using a characterization of the group algebra of the aforementioned meta-cyclic group G over the finite field F of characteristic p, where p is a positive prime such that $p \nmid 3n$.

We acknowledge that this is the first occasion where characterizing unit groups of an integral group ring contributes to identifying all integer solutions of specific symmetric polynomials with integer coefficients. This characterization not only simplifies the laborious calculations required to find a unit of the group ring RG but also translates the endeavor into finding a root of explicit polynomials over R.

सार

मुख्य शब्द: ग्रुप रिंग; इंटीग्रल ग्रुप रिंग; इकाई समूह; तुच्छ इकाइयाँ; परिमित क्षेत्र; बहुपद; सामान्य रैखिक समूह.

मान लीजिए G एक समूह है और दो तत्वों वाले चक्रीय समूह C_2 की n प्रतियों के प्रत्यक्ष उत्पाद को दर्शाता है। हम प्रत्येक सकारात्मक पूर्णांक n के लिए इकाई समूह U(ZG) के संदर्भ में अभिन्न समूह रिंग के इकाई समूह का एक लक्षण वर्णन पेश करते हैं। विशेष रूप से, हमने साबित किया कि एक समूह G के लिए। इसके अलावा, हम प्रत्येक सकारात्मक पूर्णांक n के लिए $Mx_{n}=$ $\{1 + (x_n - 1)(\sum_{i_1,i_2,\cdots,i_{n-1}} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_{n-1}} R_{i_1 i_2 \cdots i_{n-1}}) : A_n \in GL(2^k, \mathbb{Z}G), R_{i_1 i_2 \cdots i_{n-1}} \in$ Z*G* for each $i_1, i_2, \cdots, i_{n-1} \in \{0, 1\}\}$ का स्पष्ट रूप से वर्णन करते हैं। इसके द्वारा प्रत्येक Mx_n से एक मैट्रिक्स A_n जुड़ा हुआ है। प्रत्येक n = 2,3,⋯ के लिए, हम कुछ शर्तें निर्धारित करते हैं, जिसके लिए A_n प्रकार के आव्यूह $GL(2^{n-1}, ZG)$ में होता है। इसके अलावा, प्रत्येक मैट्रिक्स A_n के लिए, मैट्रिक्स A_n की प्रविष्टियों को कुछ क्रम में अनिश्चित रूप से परिवर्तित करके एक नए प्रकार के मैट्रिक्स T_n को परिभाषित किया जाता है, उसके बाद पूर्णांक गुणांक के साथ एक विशिष्ट सममित बहुपद f_n को प्रत्येक मैट्रिक्स के साथ जोड़ा जा रहा है. T_n as $f_n(X_1, X_2, \cdots, X_{2^{n-1}}) = \frac{1}{2^n}(\det(T_n) - 1)$ और इनमें से पूर्णांक समाधान का वर्णन किया गया है।

हम फॉर्म $G = \langle x,c : x^3 = 1, c^n = 1, cxc^{-1} = x^{-1} \rangle$ के मेटा-साइक्लिक समूह के लिए इकाई समूह U(ZG) को भी चिह्नित करते हैं। नतीजतन, हमने कुछ स्पष्ट बहुपदों के पूर्णांक समाधानों के सेट में तत्वों और सममित समूह S_3 और डाइसाइक्लिक समूह Dic_3 के लिए संबंधित सामान्यीकृत इकाई समूह के भीतर परिमित सूचकांक के उपसमूह में तत्वों के बीच एक-से-एक पत्राचार स्थापित करके एक परिणाम प्राप्त किया। हम विशेषता p के परिमित क्षेत्र F पर उपरोक्त मेटा-साइक्लिक समूह G के समूह बीजगणित के लक्षण वर्णन का उपयोग करते हुए प्रत्येक सकारात्मक अभाज्य $p \geq 5$ के लिए समूह रिंग ZpS_3 की प्रत्येक इकाई का भी स्पष्ट रूप से वर्णन करते हैं, जहां $p \nmid 3n$ एक सकारात्मक अभाज्य है।

हम स्वीकार करते हैं कि यह पहला अवसर है जहां एक अभिन्न समूह रिंग के इकाई समूहों को चिह्नित करने से पूर्णांक गुणांक वाले विशिष्ट सममित बहुपदों के सभी पूर्णांक समाधानों की पहचान करने में योगदान मिलता है। यह लक्षण वर्णन न केवल समूह रिंग RG की एक इकाई को खोजने के लिए आवश्यक श्रमसाध्य गणनाओं को सरल बनाता है, बल्कि R पर स्पष्ट बहुपदों की समाधान खोजने के प्रयास भी करता है।

Contents

List of Symbols

$\mathbb{Z}$	the set of integers.
$\forall x$	for all x.
$x \in X$	x belongs to X.
$G \times H$	direct product of group G and H.
$G \rtimes H$	semi-direct product of group G and H.
$\det(A)$	determinant of the square matrix A.
$\cong$	isomorphism.
$GL(n, R)$	general linear group of degree n over R.
S_n	the symmetric group of degree n.
C_n	cyclic group of order n.
R	Associative ring with unity.
RG	Group ring of group G over the ring R.
$U(RG)$	Unit group the group ring RG.
$V(RG)$	Normalized unit group the group ring RG.
F	A finite field of characteristic p.
FG	Group algebra of the group G.
$\lfloor\ \rfloor$	Greatest integer function.