SOME PROBLEMS IN
STATISTICAL THEORY OF TURBULENCE

By

Yash Paul
Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-29.

Thesis submitted to the Indian Institute of Technology, Delhi for the award of the Degree of Doctor of Philosophy in Mathematics

1967

PREFACE

During the last three decades, various attempts have been made to understand the phenomenon of turbulence. The fundamental difficulty of getting and solving a closed set of partial differential equations, representing the physical phenomenon of turbulence is still a challange to the research workers in the field. The present thesis contains some problems of the statistical theory of turbulence. It consists of seven chapters based on eight research papers, out of which five have already been published or accepted for publication in different technical journals.

In 1962, Kolmogorov modified his previous results by taking into account a circumstance, hitherto neglected, which arises directly from the assumption of the essentially accidental and random character of the mechanism of energy transfer from the larger eddies to the smaller one. Recently, Yaglom has considered the influence of fluctuations in energy dissipation on the shape of turbulence characteristics in the inertial interval and suggested some modifications in the famous "Two-thirds law" and the "five-thirds law". We have used these modified results in the first four chapters where we have discussed the problems of Radio waves and sound waves and have also obtained the relative decay rates of concentration and velocity fluctuations.

In 1965, Pao proposed a simple continuous spectral cascading process for the transfer of turbulent kinetic energy, heat and mass at large wave-numbers. In the fifth and sixth chapters we have extended Pao's analysis to obtain the spectrum functions of concentration fields of the reactant and product of reaction and turbulent multicomponent mixture with first order reactions. In the last chapter, we have discussed the problem of decay of magneto-fluid dynamic turbulence where we have obtained normal modes of decay.

While working on problems included in this thesis, we have retained the same symbols as have been used by other authors so that our results may readily be understood in the light of the results obtained in the papers referred to in the text. Consequently, some of the symbols used by us stand for different meanings in different chapters. In order to avoid any confusion, we have given the meanings of such symbols wherever necessary in the text.

CERTIFICATE

This is to certify that the thesis entitled "Some Problems in Statistical Theory of Turbulence," being submitted by Mr. Yash Paul to the Indian Institute of Technology, Delhi, for the award of the degree of Doctor of Philosophy in Mathematics, is a record of bonafide research work carried out by him. Mr. Yash Paul has worked under our guidance and supervision for the last three years. His thesis has reached the standard fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or in full, to any other university or institute for the award of any degree or diploma.

Prem Kumar

(Prem Kumar)
Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-29.

M.K. Jain

(M.K. Jain)
Head of Mathematics Department
Indian Institute of Technology
Hauz Khas, New Delhi-29.

ACKNOWLEDGEMENTS

For all that appears as my research work in the following pages, I am obliged to Dr. M.K. Jain, M.A., D. Phil., D.Sc., Head of the Mathematics Department, Indian Institute of Technology, Delhi; and Dr. Prem Kumar, M.A., Ph.D., Lecturer in Mathematics, Indian Institute of Technology, Delhi; for their kind and encouraging guidance received at every stage of my work. I am grateful to Prof. B.R. Seth, D.Sc. (London), F.N.I., Vice-Chancellor, Dibrugarh University for his kind suggestions and inspiration.

I also take this opportunity of putting on record my indebtedness to Prof. R.S. Verma, D.Sc., F.N.I., Dean, Faculty of Mathematics, University of Delhi, for his interest in my work and communicating one of my papers for publication in Proceedings of National Institute of Sciences, India. Thanks are also due to Prof. R.N. Dogra, Director, Indian Institute of Technology, Delhi, for providing the facilities to do research. I would also like to thank my numerous colleagues in the Mathematics Department, Indian Institute of Technology, Delhi, for many interesting and useful discussions that I had with them on many occasions.

In the end, I wish to thank Mr. Dev Raj Joshi for his excellent work in typing the manuscript.

Yash Paul

Indian Institute of Technology,
Hauz Khas, New Delhi-29.

(Yash Paul)
Mathematics Department

LIST OF PAPERS

1. On locally isotropic turbulence. (to be communicated)
   It includes the contents of chapter I.

2. Relative decay rates of concentration and velocity fluctuations in decaying turbulence at small Schmidt numbers. (to be communicated)
   It includes the contents of Chapter II.

3. Scattering of Radio-waves in a locally isotropic turbulent medium. Tensor, New Series. 17 (1966), 165.
   It includes the contents of chapter III.

4. On Sound-waves propagating in a locally isotropic turbulent medium. Tensor, New Series. 16 (1965), 243.
   It includes the contents of chapter IV.

5. Structure of the reactant and product of a turbulent first-order reaction at large wave-numbers.
   Proc. Nat. Inst. Sc. India, 1967 (in press).
   It includes the contents of chapter V.

6. Reactant and product concentration spectrum functions in turbulent mixing with a first-order reaction.
   Tensor, New Series 17 (1966), 308.

7. Structure of a turbulent multicomponent mixture with first-order reactions at large wave-numbers.
   (submitted for publication in J.Phys. Society of Japan)
   The papers 6 and 7 include the contents of chapter VI.

8. The decay of magneto-fluid dynamic turbulence in the presence of a uniform imposed magnetic field.
Tensor, New Series **18** (1967), 212.
It includes the contents of chapter VII.

SOME PROBLEMS IN STATISTICAL THEORY OF TURBULENCE

ABSTRACT

One of the most accepted theories of turbulence is due to Kolmogorov (C.R. Acad. Sci. USSR **30**, 1941). In this theory, he has suggested two similiarity hypotheses concenrning the local structure of turbulence at high Reynolds number. These hypotheses are based on the Richardson's idea of the existence of vortices in all possible scales $l<r<L$ between the 'external scale' L and the 'internal scale' l in the turbulent flow and it is assumed that there exists a certain uniform mechanism of energy transfer from the larger eddies to the smaller ones. Landau (Fluid Mechanics, Addison-Wesley, 1959) has shown that the mechanism of transfer of energy from the larger eddies to the smaller ones is accidental and random in character. On the basis of this property Kolmogorov (J. Fluid Mech. **13**, 1962) following the method of normal distribution discovered by Obukhov (J. Fluid Mech. **13**, 1962) has recently modified his previous theory of turbulence developed in 1940-41. Using the ideas of Kolmogorov's modified theory, Prem Kumar (Ph.D. thesis Delhi University 1962) found the longitudinal and transverse structure functions for the velocity field and the structure functions for the temperature field in the range $l<r<L$ as:

$$
(A)\quad \begin{cases}
D_{dd}(r) = C(\vec{x},t)(r\bar{\epsilon})^{2/3}\left(\frac{L}{r}\right)^{-k'}, \\
D_{nn}(r) = C(\vec{x},t)(r\bar{\epsilon})^{2/3}\left(\frac{L}{r}\right)^{-k'}\frac{3k'+8}{6}, \\
D_{T}(r) = C(\vec{x},t)\frac{\bar{N}}{(\bar{\epsilon})^{1/3}}\, r^{2/3}\left(\frac{L}{r}\right)^{-k'};
\end{cases}
$$

where $C(\vec{x},t)$ depends on the macro-structure of the turbulence. Recently, Yaglom (Soviet Physics-Doklady II, 1966) has considered the influence of fluctuations in energy dissipation on the shape of turbulence characteristics in the inertial interval and has obtained the expressions for the longitudinal and transverse structural functions for the velocity field similiar to those obtained by Kumar (1962). He has also obtained velocity spectrum E(k) as:

$$E(k) = C_2\, \overline{\epsilon^{2/3}}\, k^{-5/3} (Lk)^{-k'}.$$

Further, he has predicted an approximate value for k' viz:

k' = 0.04.

Our thesis consists of seven chapters. In the first chapter, we have applied Kolmogorov's modified hypotheses to obtain double and triple correlation functions of the velocity. In the second chapter, we have used Yaglom's velocity spectrum function to obtain the corresponding scalar spectrum functions in different ranges. By including the inertial-diffusive range, we have estimated the relative decay rates of the concentration and velocity fluctuations in decaying turbulence at Schmidt number less than 1 and for k' = 0.04.

In the third chapter we apply Kolmogorov's modified theory to the problem of scattering of Radio-waves in a locally isotropic turbulent medium. We obtain an expression for the spectral density of the structure function of refractive index which is further used to obtain the cross-section for the scattering of radio-waves. We have, also, deduced results for the Ionospheric and Tropospheric scattering. Our approach gives $\lambda^{\frac{11}{3}+k'}$ law for the frequency dependence of scattered power, where $-2/3 < k' < 4/3$. Obukhov's (1949) and Villars and Weiskopf (1955) laws are particular cases of this general law. Further, it is shown that for $-2/3 < k' < 0$, our model produces more radio scattering than Obukhov's model whereas for $0 < k' < 4/3$, radio scattering is decreased in our case.

In the fourth chapter, assuming that (i) the mean flow velocity is zero and (ii) velocity of sound is a function of temperature, we have used the functions (A) to obtain the three-dimensional spectral density of the structure function of the refractive index of sound waves propagating in a locally isotropic turbulent medium as:

$$\Phi_n(k) = (4\pi^2)^{-1} C_n^2 L^{-k'} \Gamma(k'+\tfrac{8}{3}) \sin(k'+\tfrac{2}{3})\tfrac{\pi}{2} \times k^{-(k'+\frac{11}{3})}$$

where $-2/3 < k' < 4/3$.

Further choosing the direction of propagation of the wave as x-axis and using spectral expansions, we obtain the expressions for the two-dimensional spectral densities of amplitude and phase fluctuations of the wave in the plane $x = L_o$. These expressions are also obtained on solving the equations by the method of "Smooth" perturbations. The results are compared with those obtained by Tatarski (1961). Also we obtain an expression for the effective cross-section of scattering of sound waves.

Kolmogorov's second hypothesis has been extended to case of the fluctuating scalar field by Obukhov (1949) and Corrsin (J. Appl. Phys. **22**, 1951). Following the technique of Onsager (1949) and mixing theories of Batchelor and Batchelor, Howells and Townsend (J. Fluid Mech. **5**, 1959), Corrsin (1961, 1962) estimated the form of first-order reactant and product concentration spectrum functions for various Schmidt numbers and spectral ranges. Pao (1964) modified these concepts to predict the spectra for a turbulent multicomponent mixture with isothermal first-order reaction. Recently for incompressible turbulent flow with large Reynolds and Peclet numbers, Pao (Phys. Fluids, 8, 1965) has proposed a simple continuous spectral cascading process for the transfer of turbulent kinetic energy, heat and mass at large wave-numbers. Under the restrictions (i) that the flow field and the scalar field are stationary, (ii) the Kolmogorov's hypothesis that the small-scale structure of turbulence is statistically steady and isotropic, (iii) that the small-scale structure of scalar field is statistically steady and isotropic (extension of Kolmogorov's hypothesis to the scalar field), we apply Pao's process, in the fifth chapter, to find the spectrum functions of the reactant and the product of reaction field concentration for the following spectral ranges:

(i) Universal Equilibrium range,

(ii) Inertial-convective range,

(iii) Viscous-convective and viscous-diffusive range.

In the sixth chapter, we have extended the above analysis to the case of turbulent multicomponent mixture with isothermal first-order reactions at large wave-numbers. Under the same restrictions as in chapter fifth, we have obtained spectrum functions for all the three above-mentioned spectral ranges in the following cases:

(i) Reversible first-order reaction whose stoichiometric equation is $S_1 \underset{b}{\overset{a}{\rightleftharpoons}} S_2$,

(ii) Parallel Irreversible first-order reaction whose stoichiometric equation is $S_1 \xrightarrow{a} S_2$, $S_1 \xrightarrow{b} S_3$;

(iii) Consecutive Irreversible first-order reaction whose stoichiometric equation is $S_1 \xrightarrow{a} S_2 \xrightarrow{b} S_3$;

where a and b are the reaction-rate coefficients. In this chapter, we have also considered the inertial diffusive spectral range:

$$(\epsilon / D^3)^{1/4} << k << (\epsilon / \nu^3)^{1/4}$$

with small 'Schmidt number' (ν/D) $<$ 1, and obtained the corresponding spectrum functions.

In the last chapter, the problem of homogeneous, isotropic magneto-turbulence is studied. This type of turbulence is of great importance in certain astrophysical and geophysical applications. Here we discuss how the law of decay of homogeneous magneto-turbulence is modified by the introduction of an external magnetic field. A two-point analysis of the turbulent field is carried out in the manner of Deissler (Phys. Fluids. 6, 1963). Triple correlations are neglected in order to make the problem determinate. Our basic equations include the Hall-current and we retain them till the end and solve the problem obtaining normal modes of decay.

-6-

TABLE OF CONTENTS