

# **NUMERICAL METHODS FOR SINGULAR DIFFERENTIAL EQUATIONS**

*By*

**PRAGYA JAIN**

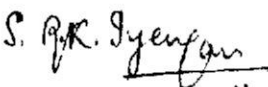
A THESIS SUBMITTED TO THE  
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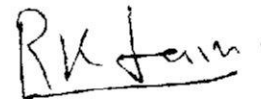
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CERTIFICATE

This is to certify that the thesis entitled "NUMERICAL METHODS FOR SINGULAR DIFFERENTIAL EQUATIONS", which is being submitted by Mrs. Pragya Jain for the award of the degree, DOCTOR OF PHILOSOPHY in Mathematics, to the Indian Institute of Technology, Delhi, is a bonafide record of research work done under our guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

  
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*Pragya Jain*  
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## SYNOPSIS

The mathematical analysis of many problems in science and engineering which have symmetry due to physical geometry in particular cylindrical and spherical geometries, lead to the singular ordinary or partial differential equations. In most cases, the analytical solution either cannot be found or has such a complicated structure that it is used only with difficulty in calculations, especially in the vicinity of the singularity. However, in the last few years the numerical methods have been successfully used for solving such problems.

In this thesis we have developed some efficient numerical methods for the numerical solution of the singular ordinary and partial differential equations. The complete discussion of the thesis has been carried out in five chapters.

### CHAPTER 1: SURVEY OF NUMERICAL METHODS FOR SINGULAR DIFFERENTIAL EQUATIONS

This chapter presents introduction, motivation with physical applications, a brief review of the numerical methods available and the summary of the work done in the present thesis.

CHAPTER 2: SINGLE-STEP METHODS FOR SECOND ORDER SINGULAR  
INITIAL VALUE PROBLEMS WITH SPHERICAL SYMMETRY

Self starting explicit single step methods of Runge-Kutta type have been derived for singular initial value problems with spherical symmetry. The methods are exact for the complementary solution of the adjoint operator of the differential equation besides having the polynomial order. The numerical results of three test problems, two linear and one nonlinear using these methods, have been compared with those obtained by using the classical explicit and implicit Runge-Kutta methods and the exact solution values to illustrate the efficiency of these methods.

CHAPTER 3: HIGHER ORDER DIFFERENCE METHODS FOR SINGULAR TWO  
POINT BOUNDARY VALUE PROBLEMS WITH SPHERICAL  
SYMMETRY

Self starting variable mesh three point difference methods of third and fifth orders are developed for the singular boundary value problems with spherical symmetry. For uniform mesh case, the third and the fifth order methods become of order four and six respectively. The methods are also exact for  $u = 1/r$ . Suitable approximations for the derivative boundary condition at the singular point are also derived. The convergence of these methods is discussed. The numerical results of four test problems, two linear and two nonlinear, obtained by using the

present methods have been compared with the exact solution to illustrate the efficiency of these methods.

CHAPTER 4\*: SPLINE FINITE DIFFERENCE METHODS FOR SINGULAR  
TWO POINT BOUNDARY VALUE PROBLEMS

The spline function approximation for a particular class of singular boundary value problems has been constructed. The consistency relation gives three point finite difference methods. These methods are of second order accuracy. The convergence of the methods has been discussed. The formulæ for cylindrical and spherical symmetry are listed explicitly. The analysis of the derivative condition at the singular point is given. The spline difference methods have been applied to four test problems including one nonlinear boundary value problem. A comparison of the numerical results has also been made with other methods as well as with the exact solution values. The numerical results demonstrate the efficiency of the new methods. Further, the spline function approximation has been used to determine the solution values in between the nodal points. The results show that the spline solutions at intermediate points in all intervals is at least of the same accuracy as the neighbouring numerical solutions.

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\*The results reported in this chapter have been published. "Spline Finite Difference Methods for Singular Two Point Boundary Value Problems", Numer. Math. 50, 363-376 (1987).

The variable mesh methods have also been developed and used to determine accurate solution near the singular point.

CHAPTER 5: SPLINE DIFFERENCE SCHEMES FOR THE CYLINDRICAL AND SPHERICAL DIFFUSION EQUATIONS

The consistency relation of the spline function approximation developed in Chapter 4 has been used to derive unconditionally stable difference schemes for the cylindrical and spherical diffusion equations. The difference schemes have order of accuracy  $(K^2 + h^2)$ . Three test problems have been solved to illustrate the efficiency of the methods. The main advantage of these methods is that whenever intermediate solutions in the spatial direction are required, they can be determined by using the numerical solutions that are already obtained and the spline function. It is found computationally that these intermediate solutions are of the same accuracy as the neighbouring numerical solutions.

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