NUMERICAL METHODS FOR SINGULAR DIFFERENTIAL EQUATIONS

By

PRAGYA JAIN

A THESIS SUBMITTED TO THE
INDIAN INSTITUTE OF TECHNOLOGY, DELHI
FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, DELHI
NEW DELHI 110016

AUGUST, 1987

CERTIFICATE

This is to certify that the thesis entitled "NUMERICAL METHODS FOR SINGULAR DIFFERENTIAL EQUATIONS", which is being submitted by Mrs. Pragya Jain for the award of the degree, DOCTOR OF PHILOSOPHY in Mathematics, to the Indian Institute of Technology, Delhi, is a bonafide record of research work done under our guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

S. R. K. Iyengar

(S.R.K. IYENGAR)

R K Jain

(R.K. JAIN)

Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-110016

ACKNOWLEDGEMENTS

It gives me great pleasure to express my profound sense of gratitude to my supervisors, Prof. S.R.K. Iyengar and Prof. R.K. Jain, Department of Mathematics, IIT Delhi, for their valuable and patient guidance and constant encouragement throughout the preparation of this thesis.

I am grateful to Prof. M.M. Chawla, Head of the Department of Mathematics, IIT Delhi, for the encouragement and help provided to me during my research work.

I am indebted to my father-in-law, Prof. M.K. Jain, without whose encouragement, this work would not have been possible.

I am also thankful to my parents for giving me inspiration at every stage of my research period. I am grateful to my husband for his patience and cooperation. I am thankful for the affectionate blessings of my mother-in-law which helped me to complete my work smoothly.

My thanks are also due to the authorities of IIT Delhi, for providing me with research facilities and financial support during the course of my research. I am also grateful to Prof. R.K. Arora, Head of the Computer Services Centre, IIT Delhi, for providing the computer facilities.

I am also thankful to my friends, Ms Vanita Verma, Ms. Sangita Rai, Rajiv Kumar, R.K. Mohanty, Vikas Bist, Sarbeswar Raut, P. Sampath Narain, R. Subramanian and Vivek Sahai, for their cooperation and valuable discussions at various stages of my work.

In the end, I would like to thank Ms. Neelam Dhody for her excellent and errorless typing.

Pragya Jain

(PRAGYA JAIN)

SYNOPSIS

The mathematical analysis of many problems in science and engineering which have symmetry due to physical geometry in particular cylindrical and spherical geometries, lead to the singular ordinary or partial differential equations . In most cases, the analytical solution either cannot be found or has such a complicated structure that it is used only with difficulty in calculations, especially in the vicinity of the singularity. However, in the last few years the numerical methods have been successfully used for solving such problems.

In this thesis we have developed some efficient numerical methods for the numerical solution of the singular ordinary and partial differential equations. The complete discussion of the thesis has been carried out in five chapters.

CHAPTER 1: SURVEY OF NUMERICAL METHODS FOR SINGULAR DIFFERENTIAL EQUATIONS

This chapter presents introduction, motivation with physical applications, a brief review of the numerical methods available and the summary of the work done in the present thesis.

CHAPTER 2: SINGLE-STEP METHODS FOR SECOND ORDER SINGULAR INITIAL VALUE PROBLEMS WITH SPHERICAL SYMMETRY

Self starting explicit single step methods of Runge-Kutta type have been derived for singular initial value problems with spherical symmetry. The methods are exact for the complementary solution of the adjoint operator of the differential equation besides having the polynomial order. The numerical results of three test problems, two linear and one nonlinear using these methods, have been compared with those obtained by using the classical explicit and implicit Runge-Kutta methods and the exact solution values to illustrate the efficiency of these methods.

CHAPTER 3: HIGHER ORDER DIFFERENCE METHODS FOR SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS WITH SPHERICAL SYMMETRY

Self starting variable mesh three point difference methods of third and fifth orders are developed for the singular boundary value problems with spherical symmetry. For uniform mesh case, the third and the fifth order methods become of order four and six respectively. The methods are also exact for $u = 1/r$. Suitable approximations for the derivative boundary condition at the singular point are also derived. The convergence of these methods is discussed. The numerical results of four test problems, two linear and two nonlinear, obtained by using the

present methods have been compared with the exact solution to illustrate the efficiency of these methods.

CHAPTER 4*: SPLINE FINITE DIFFERENCE METHODS FOR SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS

The spline function approximation for a particular class of singular boundary value problems has been constructed. The consistency relation gives three point finite difference methods. These methods are of second order accuracy. The convergence of the methods has been discussed. The formulae for cylindrical and spherical symmetry are listed explicitly. The analysis of the derivative condition at the singular point is given. The spline difference methods have been applied to four test problems including one nonlinear boundary value problem. A comparison of the numerical results has also been made with other methods as well as with the exact solution values. The numerical results demonstrate the efficiency of the new methods. Further, the spline function approximation has been used to determine the solution values in between the nodal points. The results show that the spline solutions at intermediate points in all intervals is at least of the same accuracy as the neighbouring numerical solutions.

---

*The results reported in this chapter have been published. "Spline Finite Difference Methods for Singular Two Point Boundary Value Problems", Numer. Math. 50, 363-376(1987).

The variable mesh methods have also been developed and used to determine accurate solution near the singular point.

CHAPTER 5: SPLINE DIFFERENCE SCHEMES FOR THE CYLINDRICAL AND SPHERICAL DIFFUSION EQUATIONS

The consistency relation of the spline function approximation developed in Chapter 4 has been used to derive unconditionally stable difference schemes for the cylindrical and spherical diffusion equations. The difference schemes have order of accuracy $(K^2 + h^2)$. Three test problems have been solved to illustrate the efficiency of the methods. The main advantage of these methods is that whenever intermediate solutions in the spatial direction are required, they can be determined by using the numerical solutions that are already obtained and the spline function. It is found computationally that these intermediate solutions are of the same accuracy as the neighbouring numerical solutions.

CONTENTS

		PAGE NO
	SYNOPSIS	i-iv
CHAPTER 1:	SURVEY OF NUMERICAL METHODS FOR SINGULAR DIFFERENTIAL EQUATIONS	1-42
1.1	Introduction	1
1.2	Initial Value Problems	6
1.3	Boundary Value Problems	19
1.4	Conclusions	34
1.5	Description of the Thesis	35
	Tables	39
CHAPTER 2:	SINGLE-STEP METHODS FOR SECOND ORDER SINGULAR INITIAL VALUE PROBLEMS WITH SPHERICAL SYMMETRY	43-81
2.1	Introduction	43
2.2	Derivation of Methods	44
	2.2.1 Two stage method	45
	2.2.2 Three stage method	50
	2.2.3 Four stage method	57
2.3	Application of Classical Runge-Kutta-Nyström Methods	68
2.4	Numerical Results	72
2.5	Conclusion	74
	Tables	75
	References	81

CHAPTER 3: HIGHER ORDER DIFFERENCE METHODS FOR SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS WITH SPHERICAL SYMMETRY 82-119

3.1 Introduction 82

3.2 Derivation of Methods 85

3.2.1 Third order method 87

3.2.2 Fifth order method 91

3.3 Approximation for the Boundary Condition $u'(0) = 0$ 98

3.4 Convergence of the Methods 101

3.5 Numerical Comparison 108

3.6 Conclusion 111

Tables 112

References 118

CHAPTER 4: SPLINE FINITE DIFFERENCE METHODS FOR SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS 120-176

4.1 Introduction 120

4.2 Splines and Finite Difference Methods 122

4.3 Convergence of the Spline Difference Methods 137

4.4 Treatment of the Boundary Condition $u'(0) = 0$ 144

4.5 Computational Implementation 146

4.6 Conclusion 153

Tables 154

References 175

CHAPTER 5: SPLINE DIFFERENCE SCHEMES FOR THE CYLINDRICAL AND SPHERICAL DIFFUSION EQUATIONS 177-202

5.1 Introduction 177

5.2 Difference Scheme for Cylindrical Diffusion Equation 179

5.3 Difference Scheme for the Spherical Diffusion Equation 190

5.4 Conclusion 196

Tables 197

References 202

BIBLIOGRAPHY 203-212