

# KERNEL AND IMAGE OF DERIVATIONS OF THE POLYNOMIAL ALGEBRA

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**KERNEL AND IMAGE OF DERIVATIONS OF THE  
POLYNOMIAL ALGEBRA**

by

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*Submitted*

*in fulfillment of the requirements of the degree of Doctor of Philosophy  
to the*




**Indian Institute of Technology Delhi  
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*Dedicated to  
my family*

# Certificate

This is to certify that the thesis entitled **KERNEL AND IMAGE OF DERIVATIONS OF THE POLYNOMIAL ALGEBRA** submitted by **Ms. Sakshi Gupta** to **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi  
December 2023

  
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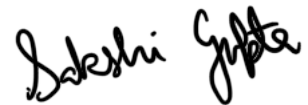
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Sakshi Gupta

# Abstract

Let  $K$  be a field of characteristic zero and  $K[X] = K[x_1, x_2, \dots, x_n]$  be the polynomial algebra in  $n$  variables over  $K$ . The main emphasis of this thesis is to study the kernel and the image of  $K$ -derivations and  $K\mathcal{E}$ -derivations defined over  $K[X]$ .

We started by reviewing some literature and known results on  $K$ -derivations, which play a crucial role in the work done in this thesis. This thesis is divided into two parts. In the first part, we study the field of rational constants and Darboux polynomials of a generalized cyclotomic derivation  $d$  of  $K[X]$ . Further, it is shown that  $d$  is without Darboux polynomials if and only if  $K(X)^d = K$ . The result is also studied in the tensor product of polynomial algebras.

In the second part of the thesis, we study the image of linear  $K$ -derivations and  $K\mathcal{E}$ -derivations of  $K[X]$ . We show that if the eigenvalues of the associated matrix of a linear  $K$ -derivation  $d$  are linearly independent over  $\mathbb{N}_0$ , then the image of  $d$  is a maximal ideal in  $K[X]$ . Further, we show that the LFED conjecture holds for certain linear  $K$ -derivations of  $K[x_1, x_2, x_3, x_4]$ . Furthermore, we assume certain conditions on the eigenvalues corresponding to the linear  $K$ -endomorphism  $I - \delta$  of  $K[x_1, x_2, x_3, x_4]$ , where  $\delta$  is a linear  $K\mathcal{E}$ -derivation of  $K[x_1, x_2, x_3, x_4]$  and prove that the LFED conjecture is valid for linear  $K\mathcal{E}$ -derivations of  $K[x_1, x_2, x_3, x_4]$ .

Moreover, we prove that for a linear  $K$ -derivation  $d$  of  $K[X]$  and the maximal ideal  $\mathfrak{m} = (x_1, x_2, \dots, x_n)$  of  $K[X]$ , if  $d(\mathfrak{m})$  is a Mathieu-Zhao subspace of  $K[X]$ , then the image of every  $\mathfrak{m}$ -primary ideal under  $d$  forms a Mathieu-Zhao subspace of  $K[X]$ . Additionally, we observe that the image of all monomial ideals under the  $K$ -derivation  $d = f\partial_{x_1}$  of  $K[X]$ , for  $f \in K[X]$  forms an ideal of  $K[X]$ . Finally, we prove that the image of certain monomial ideals under a linear locally nilpotent  $K$ -derivation of  $K[x_1, x_2, x_3]$  defined by  $d = x_2\partial_{x_1} + x_3\partial_{x_2}$  forms a Mathieu-Zhao subspace.

## सार

कोई क्षेत्र जिसका विशेषता शून्य है और  $K[X] = K[x_1, x_2, \dots, x_n]$  एक ऐसा बहुपद बीजगणित है जो  $K$  के ऊपर  $n$  चरणों में है। इस थीसिस का मुख्य उद्दीपन  $K[X]$  पर परिभाषित  $K$ -डेरिवेशन और  $K_E$ -डेरिवेशन के कर्णल और छवि का अध्ययन करना है।

हमने  $K$ -डेरिवेशन पर कुछ साहित्य और जाने गए परिणामों की समीक्षा करके शुरुआत की, जो इस थीसिस में किए गए काम में एक महत्वपूर्ण भूमिका निभाते हैं। यह थीसिस दो भागों में विभाजित है। पहले भाग में, हम एक सामान्यकृत साइक्लोटॉमिक डेरिवेशन  $d$  के रेशनल स्थिरांक और डार्लू पॉलिनोमियल्स का अध्ययन करते हैं। इसके अतिरिक्त, दिखाया गया है कि  $d$  डार्लू पॉलिनोमियल्स के बिना है अगर और केवल अगर  $K(X)^d = K$ । इस परिणाम का अध्ययन पॉलिनोमियल बीजगणितों के टैंसर प्रोडक्ट में भी किया गया है।

थीसिस के दूसरे भाग में, हम  $K[X]$  के रैखिक  $K$ -डेरिवेशन और  $K_E$ -डेरिवेशन की छवि का अध्ययन करते हैं। हम दिखाते हैं कि यदि एक रैखिक  $K$ -डेरिवेशन  $d$  के संबंधित मैट्रिक्स के इजनवैल्यूज  $N_0$  के ऊपर रैखिक रूप से स्वतंत्र हैं, तो  $d$  की छवि  $K[X]$  में एक अधिकतम आदान है। इसके अतिरिक्त, हम दिखाते हैं कि कुछ ऐसे रैखिक  $K$ -डेरिवेशनों के लिए LFED अनुमान सत्य है जो  $K[x_1, x_2, x_3, x_4]$  के हैं। इसके अतिरिक्त, हम  $K[x_1, x_2, x_3, x_4]$  के रैखिक  $K$ -अंतःप्रवर्तन  $l - \delta$  के लिए इजनवैल्यूज के साथ कुछ शर्तें मानते हैं, जहां  $\delta$   $K[x_1, x_2, x_3, x_4]$  का एक रैखिक  $K_E$ -डेरिवेशन है, और सिद्ध करते हैं कि LFED अनुमान रैखिक  $K_E$ -डेरिवेशनों के लिए सत्य है।

इसके अतिरिक्त, हम सिद्ध करते हैं कि  $K[X]$  के एक रैखिक  $K$ -डेरिवेशन  $d$  और  $m = (x_1, x_2, \dots, x_n)$  का अधिकतम आदान, यदि  $d(m)$   $K[X]$  का एक Mathieu-Zhao उपक्षेत्र है, तो  $d$  के अधीन सभी  $m$ -प्राथमिक आदानों की छवि  $K[X]$  का एक Mathieu-Zhao उपक्षेत्र बनाती है। इसके अतिरिक्त, हम देखते हैं कि  $K$ -डेरिवेशन  $d = f\partial_{x_1}$  के लिए, जहां  $f \in K[X]$ ,  $K[X]$  के सभी मॉनोमियल आदानों की छवि एक आदान बनाती है। अंत में, हम सिद्ध करते हैं कि  $K[x_1, x_2, x_3]$  के एक रैखिक स्थानीय शून्यस्थान  $K$ -डेरिवेशन  $d = x_2\partial_{x_1} + x_3\partial_{x_2}$  के नीचे कुछ मॉनोमियल आदानों की छवि एक Mathieu-Zhao उपक्षेत्र बनाती है।

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# List of Symbols

<b>Symbol</b>	<b>Meaning</b>
$\forall$	for all
$\exists$	there exists
$\in$	belongs to
$\subseteq$	subset or equal
$\subsetneq$	proper subset
$\cup, \cap$	union, intersection
$\sqcup$	disjoint union
$\emptyset$	empty set
$\mathbb{Z}$	the ring of integers
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the real line
$\mathbb{C}$	the field of complex numbers
$K$	a field of characteristic zero
$A$	a commutative $K$ -algebra
$K[X]$	the polynomial algebra in $n$ variables over $K$
$K(X)$	the field of fractions of $K[X]$
$d$	a $K$ -derivation of $A$
$\text{Der}(A, M)$	set of all derivations from $A$ to $M$
$\delta$	a $K\mathcal{E}$ -derivation of $A$
$\partial_x$	the standard partial derivative w.r.t. the variable $x$

$\text{Ker}(d) = A^d$	Kernel of the $K$ -derivation $d$ of $A$
$\text{Im}(d)$	Image of the $K$ -derivation $d$ of $A$
$\mathfrak{p}$	a prime ideal of $K[X]$
$\mathfrak{m}$	a maximal ideal of $K[X]$
$\mathcal{M}$	a $K$ -linear subspace of $A$
$\otimes_K$	tensor product of $K$ -algebra
$\oplus$	direct sum
$r(\mathcal{M})$	radical of the $K$ -subspace $\mathcal{M}$
$[x]$	greatest interger function of $x$ for $x \in \mathbb{R}$
$\square$	end of a proof