

APPROXIMATION OF NONLINEAR TIME-FRACTIONAL PDES USING FINITE ELEMENT METHODS

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APPROXIMATION OF NONLINEAR TIME-FRACTIONAL PDES USING FINITE ELEMENT METHODS

by

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Dedicated to
My Family

Certificate

This is to certify that the thesis entitled “**Approximation of nonlinear time-fractional PDEs using finite element methods**” submitted by “**Mr. Dileep Kumar**” to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

Fractional order partial differential equations (PDEs) can be characterized as an extended form of integer-order PDEs, specially designed to model certain real-world phenomena that can not be described well by integer-order PDEs. Some well-known applications of fractional PDEs include anomalous diffusion process, fractals, automatic control systems with feedback, traffic model, population dynamics, and so on [9, 22, 31, 62].

This thesis aims to find numerical solutions of scalar and coupled time-fractional nonlinear diffusion equations. We develop efficient numerical schemes involving finite element spatial discretization. For time discretization, we employ several popular approximations to the fractional derivatives existing in the literature. A significant contribution of this thesis includes establishing *a priori* bounds and convergence estimates in terms of a newly developed discrete fractional Grönwall type inequalities. In this thesis, we solve many time-fractional nonlinear scalar and coupled diffusion equations by establishing various novel and effective numerical schemes. Firstly, we propose a first-order accurate Grünwald-Letnikov-Galerkin finite element scheme for solving the time-fractional nonlinear diffusion equation. A discrete fractional Grönwall inequality is derived and employed to prove *a priori* bounds and convergence estimates in both $L^2(\Omega)$ and $H^1(\Omega)$ norms.

Next, we present two new fully-discrete schemes to obtain the numerical solution of

the coupled time-fractional nonlinear diffusion system. *A priori* bounds and *a priori* error estimates are obtained for the presented schemes. Further, we reconsider the time-fractional nonlinear diffusion equation and establish a fractional Crank-Nicolson-Galerkin finite element scheme which is known to recover second-order accuracy. Well-posedness and convergence results are proved for the fully-discrete solution. Due to the advantage of the fractional Crank-Nicolson scheme over $L1$, we revisit the coupled time-fractional nonlinear diffusion system and propose two new second-order accurate fully-discrete schemes for solving this coupled system. Well-posedness results and *a priori* error estimates for both schemes are obtained in $L^2(\Omega)$ norm. Finally, we analyze the numerical solution of the time-fractional nonlinear delay diffusion equation by presenting two linearized fully-discrete schemes. In each chapter, all our theoretical results are validated via one and two-dimensional numerical examples.

सारांश

भिन्नात्मक क्रम आंशिक अंतर समीकरणों को पूर्णांक-क्रम आंशिक अंतर समीकरणों के विस्तारित रूप में चित्रित किया जा सकता है, विशेष रूप से कुछ वास्तविक दुनिया की घटनाओं को मॉडल करने के लिए डिज़ाइन किया गया है, ऐसी घटनाएँ जिन्हें पूर्णांक-क्रम आंशिक अंतर समीकरणों द्वारा अच्छी तरह से वर्णित नहीं किया जा सकता है। भिन्नात्मक क्रम आंशिक अंतर समीकरणों के कुछ प्रसिद्ध उपयोगों में विसंगति प्रसार प्रक्रिया, भग्न, प्रतिक्रिया के साथ स्वचालित नियंत्रण प्रणाली, ट्रेफिक मॉडल, जनसंख्या की गतिशीलता, आदि शामिल हैं [9, 22, 31, 62]।

इस थीसिस का उद्देश्य स्केलर और कपल्ड समय-भिन्नात्मक अरेखीय प्रसार समीकरणों के संख्यात्मक हलों को निकालना है। हम परिमित तत्व स्थानिक असंततकरण के साथ कुशल संख्यात्मक योजनाएँ विकसित करते हैं। समय के असंततकरण हम साहित्य में विद्यमान भिन्नात्मक व्युत्पन्न के लिए कई लोकप्रिय समीपता का उपयोग करते हैं।

इस थीसिस के एक महत्वपूर्ण योगदान में एक नई विकसित असतत भिन्नात्मक ग्रोनवॉल प्रकार की असमानताओं के संदर्भ में एक प्राथमिक सीमा और अभिसरण अनुमान स्थापित करना शामिल है। इस थीसिस में हम विभिन्न नई और प्रभावी संख्यात्मक योजनाओं की स्थापना करके विभिन्न समय के भिन्नात्मक अरेखीय स्केलर और कपल्ड प्रसार समीकरणों को हल करते हैं।

सबसे पहले हम समय-भिन्नात्मक अरेखीय प्रसार समीकरण को हल करने के लिए प्रथम-क्रम की सटीक ग्रुन्वाल्ड-लेटनिकोव-गैलेर्किन परिमित तत्व योजना का प्रस्ताव करते हैं। एक असतत भिन्नात्मक ग्रोनवॉल असमानता साबित होती है और दोनों मानदंडों $L_2(\Omega)$ और $H_1(\Omega)$ में एक प्राथमिक सीमा और अभिसरण अनुमान साबित करने के लिए प्रयुक्त किया जाता है। आगे हम दो नए पूरी तरह से असतत योजनाओं को कपल्ड समय-भिन्नात्मक अरेखीय प्रसार तंत्र के संख्यात्मक समाधान प्राप्त करने के लिए प्रस्तुत करते हैं।

प्रस्तुत योजनाओं के लिए एक प्राथमिक सीमा और एक प्राथमिक त्रुटि अनुमान प्राप्त किए जाते हैं। इसके अलावा हम समय-भिन्नात्मक अरेखीय प्रसार समीकरण पर पुनर्विचार करते हैं और एक भिन्नात्मक क्रैंक-निकोलसन-गैलेर्किन

परिमित तत्व योजना की स्थापना करते हैं जिसे द्वितीय-क्रम सटीकता को पुनर्प्राप्त करने के लिए जाना जाता है। पूरी तरह से असतत समाधान के लिए वेलपोसडनेस और अभिसरण परिणाम साबित होते हैं। L_1 के ऊपर फ्रैक्शनल क्रैंक-निकोलसन स्कीम के लाभ के कारण हम कपलड समय-भिन्नात्मक अरेखीय प्रसार तंत्र को फिर से देखते हैं और इस कपलड तंत्र को हल करने के लिए दो नए द्वितीय-क्रम सटीक पूर्ण-डिस्क्रीट योजनाओं का प्रस्ताव करते हैं। दोनों योजनाओं के लिए वेलपोसडनेस परिणाम और एक प्राथमिक त्रुटि अनुमान $L_2(\Omega)$ मानदंड में प्राप्त किए जाते हैं। अंत में हम दो रैखिक पूर्ण-असतत योजनाओं को प्रस्तुत करके समय-भिन्नात्मक अरेखीय देरी प्रसार समीकरण के संख्यात्मक समाधान का विश्लेषण करते हैं।

प्रत्येक अध्याय में हमारे सभी सैद्धांतिक परिणाम एक और दो-आयामी आकार उदाहरणों के माध्यम से मान्य होते हैं।

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List of Symbols

Symbol Meaning

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^+$	Set of positive real numbers
$\forall x$	For all x
$x \in X$	x is a member of X
Δ	Laplacian
∇	Gradient
${}_a D_t^\alpha$	Grünwald-Letnikov fractional derivative
${}^R D_t^\alpha$	Riemann-Liouville fractional derivative
${}^C D_t^\alpha$	Caputo fractional derivative
${}^C D_{\Delta t}^\alpha$	Discrete fractional operator corresponding to Caputo fractional derivative approximation
${}^R D_{\Delta t}^\alpha$	Discrete fractional operator corresponding to Riemann-Liouville fractional derivative approximation

$\Gamma(\cdot)$	Gamma function
$E_\alpha(\cdot)$	Single parameter Mittag-Leffler function
$E_{\alpha,\beta}(\cdot)$	Two parameter Mittag-Leffler function
X_h	Finite dimensional space
R_h	Ritz projection operator
C^α	Space of Hölder continuous functions with exponent α
C^n	Space of n -times continuous differentiable functions
$C_0^\infty(\Omega)$	Space of infinitely differentiable functions with compact support in Ω
$H_0^k(\Omega)$	Trace zero elements of Sobolev space $H^k(\Omega)$
$\ w\ _m$	$\left(\sum_{0 \leq \alpha \leq m} \left\ \frac{\partial^\alpha w}{\partial x^\alpha} \right\ ^2 \right)^{1/2}$