

# ISOTROPY GROUPS OF NON-SIMPLE DERIVATIONS OF POLYNOMIAL ALGEBRAS

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# Isotropy Groups of Non-Simple Derivations of Polynomial Algebras

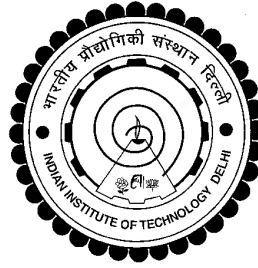
by

**Himanshu**

**Department of Mathematics**

*Submitted*

*in fulfillment of the requirements of the degree of Doctor of Philosophy  
to the*



**Indian Institute of Technology Delhi  
July 2025**

*Dedicated to  
my family*

# Certificate

This is to certify that the thesis entitled **Isotropy Groups of Non-Simple Derivations of Polynomial Algebras** submitted by **Mr. Himanshu** to **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi  
July 2025

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**Associate Professor**  
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# Abstract

Let  $K$  denote a field of characteristic zero containing  $\mathbb{Q}$ ,  $K^* = K \setminus \{0\}$ , and  $K[X] = K[x_1, \dots, x_n]$  be the polynomial algebra in  $n$  variables over  $K$ . This thesis focuses on finding the isotropy groups of non-simple derivations of  $K[X]$ .

We start by reviewing some literature and known results on  $K$ -derivations, which play a crucial role in the work done in this thesis. In Chapter 2, we study the isotropy groups of non-simple derivations of the form  $x_1^m x_2^u D$ , where  $m, u \geq 0$ , and  $D$  is a simple derivation of  $K[x_1, x_2]$ . For  $m, u \geq 1$ , we prove that the isotropy group of  $x_1^m x_2^u D$  is isomorphic to the product of two finite cyclic groups. Further, it is shown that isotropy groups of derivations  $x_1^m D$  and  $x_2^u D$  are finite, where  $m, u$  are positive integers.

In Chapter 3, we study the isotropy group of the Lotka-Volterra derivation of  $K[X]$ , i.e., a derivation  $d$  of the form  $d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$ , where  $C_i \in K$  and  $1 \leq i \leq n$ , with  $x_0 = x_n$  and  $x_{n+1} = x_1$ . If  $n = 3$  or  $n \geq 5$ , we have shown that the isotropy group of  $d$  is finite. However, for  $n = 4$ , it is observed that the isotropy group of  $d$  need not be finite. Indeed, for  $C_i = -1$ , we identify an infinite set of automorphisms in the isotropy group of  $d$ . Moreover, for  $n \geq 3$ , and  $C_i = 1$ , we have shown that the isotropy group of  $d$  is isomorphic to the dihedral group of order  $2n$ . Furthermore, if  $C_i = 0$  for all  $i$ , then the isotropy group of  $d$  is isomorphic to a cyclic group of order  $n$ .

In Chapter 4, we study the isotropy group of the Jouanolou derivation, a well-known monomial derivation of  $K[X]$  of the form

$$d(x_1) = x_2^s, \quad d(x_2) = x_3^s, \quad \dots, \quad d(x_{n-1}) = x_n^s, \quad d(x_n) = x_1^s,$$

where  $s \geq 1$ . For  $s \geq 2$ , we show that its isotropy group is generated by the sets  $\Gamma_n$ , consisting of cyclic permutations of the variables  $x_i$ 's for  $1 \leq i \leq n$ , and  $\Delta_n$ , consisting of automorphisms

isomorphic to the multiplicative group of roots of unity of order  $s^n - 1$ . In contrast, the isotropy group of the corresponding factorizable derivation is generated solely by  $\Gamma_n$ .

Finally, in the last chapter, we summarize the thesis and propose future work directions.

# सार

माना कि  $K$  शून्य अभिलाक्षणिकता वाला एक क्षेत्र है जिसमें  $\mathbb{Q}$  समाहित है,  $K^* = K \setminus \{0\}$ , और  $K[X] = K[x_1, x_2, \dots, x_n]$ ,  $K$  पर  $n$  चरों में बहुपदीय बीजगणित है। यह शोधप्रबंध  $K[X]$  के असंक्षिप्त अवकलनों के समस्थानिक समूहों को खोजने पर केंद्रित है।

हम  $K$ -अवकलनों पर कुछ साहित्य और ज्ञात परिणामों की समीक्षा करके प्रारंभ करते हैं, जो इस शोधप्रबंध में किए गए कार्य में महत्वपूर्ण भूमिका निभाते हैं। अध्याय 2 में, हम  $x_1^m x_2^u D$  रूप के असंक्षिप्त अवकलनों के समस्थानिक समूहों का अध्ययन करते हैं, जहाँ  $m, u \geq 0$ , और  $D, K[x_1, x_2]$  का एक संक्षिप्त अवकलन है।  $m, u \geq 1$  के लिए, हम सिद्ध करते हैं कि  $x_1^m x_2^u D$  का समस्थानिक समूह दो परिमित चक्रीय समूहों के गुणनफल के समरूप होता है। साथ ही, यह दिखाया गया है कि  $x_1^m D$  और  $x_2^u D$  के समस्थानिक समूह परिमित होते हैं, जहाँ  $m, u$  धनात्मक पूर्णांक हैं।

अध्याय 3 में, हम  $K[X]$  के लोटका-वोल्टेरा अवकलन के समस्थानिक समूह का अध्ययन करते हैं, अर्थात् एक अवकलन  $d$  जिसका रूप  $d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$  है, जहाँ  $C_i \in K$  और  $1 \leq i \leq n$ , तथा  $x_0 = x_n$  और  $x_{n+1} = x_1$ । यदि  $n = 3$  या  $n \geq 5$  है, तो हमने दिखाया है कि  $d$  का समस्थानिक समूह परिमित होता है। हालाँकि,  $n = 4$  के लिए, यह देखा गया है कि  $d$  का समस्थानिक समूह परिमित नहीं होना चाहिए। वास्तव में,  $C_i = -1$  के लिए, हम  $d$  के समस्थानिक समूह में अनंत स्वरूपसमष्टियों की पहचान करते हैं। इसके अलावा,  $n \geq 3$  और  $C_i = 1$  के लिए, हमने दिखाया है कि  $d$  का समस्थानिक समूह क्रम  $2n$  के द्विभाजक समूह  $S_{2n}$  के समरूप होता है। साथ ही, यदि सभी  $i$  के लिए  $C_i = 0$  है, तो  $d$  का समस्थानिक समूह क्रम  $n$  के चक्रीय समूह के समरूप होता है।

अध्याय 4 में, हम जौनोलू अवकलन के समस्थानिक समूह का अध्ययन करते हैं, जो  $K[X]$  का एक प्रसिद्ध एकपदीय अवकलन है, जिसका रूप है:

$$d(x_1) = x_2^s, \quad d(x_2) = x_3^s, \quad \dots \quad d(x_{n-1}) = x_n^s, \quad d(x_n) = x_1^s,$$

जहाँ  $s \geq 1$ ।  $s \geq 2$  के लिए, हम दिखाते हैं कि इसका समस्थानिक समूह  $\Gamma_n$  और  $\Delta_n$  द्वारा उत्पन्न होता है। इसके विपरीत, संगत गुणनखंडनीय अवकलन का समस्थानिक समूह केवल  $\Gamma_n$  द्वारा उत्पन्न होता है।

अंतिम अध्याय में, हम शोधप्रबंध का सारांश प्रस्तुत करते हैं और भविष्य के कार्य की दिशाओं का प्रस्ताव रखते हैं।



# Contents

|  |            |
|--|------------|
| <b>Certificate</b>   | <b>i</b>   |
| <b>Acknowledgments</b>   | <b>iii</b> |
| <b>Abstract</b>  | <b>v</b>   |
| <b>List of Symbols</b>   | <b>xi</b>  |
| <b>Introduction</b>  | <b>1</b>   |
| <b>1 Preliminaries</b>   | <b>5</b>   |
| 1.1 Derivation . . . . .   | 5          |
| 1.2 Homogeneous derivations . . . . .  | 9          |
| 1.3 Simple derivations . . . . .   | 10         |
| 1.4 Isotropy groups . . . . .  | 14         |
| <b>2 Isotropy groups of non-simple derivations of <math>K[x_1, x_2]</math></b> | <b>27</b>  |
| 2.1 Introduction . . . . .   | 27         |
| 2.2 Preliminaries . . . . .  | 27         |
| 2.3 Monomial multiple of Shamsuddin derivation . . . . .                       | 29         |
| 2.4 Monomial multiple of Quadratic derivation . . . . .                        | 36         |
| 2.5 Monomial multiple of higher degree derivation . . . . .                    | 41         |
| <b>3 Isotropy groups of Lotka-Volterra derivations of <math>K[X]</math></b>    | <b>55</b>  |
| 3.1 Introduction . . . . .   | 55         |

---

|          |  |            |
|----------|--|------------|
| 3.2      | Isotropy group of Lotka-Volterra derivation of $K[x_1, x_2, x_3]$ . . . . .      | 56         |
| 3.3      | Isotropy group of Lotka-Volterra derivation of $K[x_1, x_2, x_3, x_4]$ . . . . . | 70         |
| 3.4      | Isotropy group of Lotka-Volterra derivation of $K[x_1, \dots, x_n]$ . . . . .    | 86         |
| <b>4</b> | <b>Isotropy groups of Jouanolou derivations of <math>K[X]</math></b>             | <b>95</b>  |
| 4.1      | Introduction . . . . .   | 95         |
| 4.2      | Isotropy group of Jouanolou derivations . . . . .                                | 97         |
| 4.3      | Factorizable derivations . . . . .   | 100        |
| <b>5</b> | <b>Summary and future work</b>   | <b>107</b> |
| 5.1      | Summary . . . . .  | 107        |
| 5.2      | Future work . . . . .  | 107        |
|          | <b>List of Publications</b>  | <b>109</b> |
|          | <b>Bibliography</b>  | <b>111</b> |
|          | <b>Curriculum Vitae</b>  | <b>115</b> |

# List of Symbols

| <b>Symbol</b>                 | <b>Meaning</b>  |
|-------------------------------|---|
| $\forall$                     | for all   |
| $\exists$                     | there exists  |
| $\in$                         | belongs to  |
| $\subseteq$                   | subset or equal   |
| $\subsetneq$                  | proper subset   |
| $\cup$                        | union   |
| $\cap$                        | intersection  |
| $\sqcup$                      | disjoint union  |
| $\emptyset$                   | empty set   |
| $\mathbb{N}$                  | the set of natural numbers                              |
| $\mathbb{N}_0$                | $\mathbb{N} \cup \{0\}$                                 |
| $\mathbb{Z}$                  | the ring of integers                                    |
| $\mathbb{Z}_n$                | the ring of integers modulo $n$                         |
| $\mathbb{Q}$                  | the field of rational numbers                           |
| $K$                           | a field of characteristic zero                          |
| $K^*$                         | $K \setminus \{0\}$                                     |
| $A$                           | a commutative $K$ -algebra                              |
| $K[X]$                        | the polynomial algebra in $n$ variables over $K$        |
| $K(X)$                        | the field of fractions of $K[X]$                        |
| $d$                           | a $K$ - derivation of $A$                               |
| $\text{Der}(A)$               | set of all derivations from $A$ to $A$                  |
| $\frac{\partial}{\partial x}$ | the standard partial derivative w.r.t. the variable $x$ |

|  |   |
|--|---|
| $\text{Aut}(K[X])_d$                   | the isotropy group of $d$ of $K[X]$             |
| $\text{deg}_{x_i}(f(x_1, \dots, x_n))$ | $x_i$ -degree of $f(x_1, \dots, x_n)$ in $K[X]$ |
| $\square$                              | end of a proof                                  |