NONREFLECTING BOUNDARY OPERATORS IN 2D/3D

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Certificate

This is to certify that the thesis titled **"Nonreflecting boundary operators in 2D/3D"**, submitted by **Ms. Samardhi**, to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy**, is a bonafide record of the research work done by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Abstract

This dissertation addresses the challenges involved in choosing hyperrectangular computational domains for the numerical solution of wave equations that are originally defined over $\mathbb{R}^d,\ d = 2, 3$. These challenges mainly arise due to the presence of corners in the computational domain and nonlocal nature of the boundary maps. Let us acknowledge that the existing convolution quadrature (CQ) approach for fractional operators, while considered a golden standard for accuracy, is computationally expensive and thus presents a significant challenge. This research aims to design numerical schemes which can provide a viable trade-off between computational complexity and accuracy.

To achieve this, we introduce a Padé approximant based rational approximation for certain fractional operators, effectively handling the corners and yielding a temporally local approximation for the nonlocal boundary maps with time-step-independent complexity. Furthermore, we present several tests to compare the performance of our novel padé approach with the existing CQ approach. Additionally, we consider spatially local boundary conditions (BCs) obtained using high-frequency approximation of nonlocal boundary maps. Let us note that the use of the high-frequency BCs may impact accuracy but they are easier to implement and affords comparatively lower computational complexity which makes them attractive.

For the spatial discretization, we use a Legendre-Galerkin spectral method with a new boundary adapted basis which ensures that the resulting linear system is banded. A compatible boundary-lifting procedure is also presented which accommodates the segments as well as the corners on the boundary. Temporal discretization is then addressed using one-step time marching schemes, namely, the backward-differentiation formula of order 1 (BDF1) and the trapezoidal rule (TR). Note that an unsuitable temporal discretization of the nonlocal operators may destroy the overall stability of the numerical scheme in this approach.

Alternatively, we provide a approach wherein the initial boundary-value problem is first discretized in space, incorporating the boundary conditions, to yield a time-continuous dynamical system. This system is then discretized in time using various time-stepping methods, resulting a fully discrete system. This approach facilitates the use of higher order time-marching schemes, potentially improving the accuracy of the overall system. Numerical tests are presented to demonstrate the efficacy of our novel method and to confirm the order of convergence empirically.

सार

यह शोध प्रबंध हाइपररेक्टेंगुलर कम्प्यूटेशनल डोमेन को चुनने में शामिल चुनौतियों का समाधान करता है तरंग समीकरणों के संख्यात्मक समाधान के लिए डोमेन जो मूल रूप से Rd, d = 2, 3 पर परिभाषित हैं। ये चुनौतियाँ मुख्य रूप से कम्प्यूटेशनल डोमेन में कोनों की उपस्थिति और सीमा मानचित्रों की गैर-स्थानीय प्रकृति के कारण उत्पन्न होती हैं । आइए हम स्वीकार करें कि फ्रैक्शनल ऑपरेटरों के लिए मौजूदा कनवल्शन क्वाडरेचर (सीक्यू) दृष्टिकोण, सटीकता के लिए एक सुनहरा मानक माना जाता है लेकिन यह कम्प्यूटेशनल रूप से महंगा है यह इस प्रकार एक महत्वपूर्ण चुनौती प्रस्तुत करता है। इस शोध का उद्देश्य डिजाइन करना है संख्यात्मक योजनाएं जो कम्प्यूटेशनल जटिलता और शुद्धता के बीच एक व्यवहार्य व्यापार-बंद प्रदान कर सकती है।

इसे प्राप्त करने के लिए, हम कुछ भिन्नात्मक ऑपरेटरों के लिए पैडे सन्निकटन आधारित तर्कसंगत सन्निकटन प्रस्तुत करते हैं जो कोनों को प्रभावी ढंग से संभालता है और अस्थायी रूप से स्थानीय सन्निकटन उत्पन्न करता है। यह गैर-स्थानीय सीमा मानचित्रों के लिए एक समय-चरण-स्वतंत्र जटिलता देता है। इसके अलावा, हम प्रस्तुत करते हैं मौजूदा सीक्यू के साथ हमारे नए पैड दृष्टिकोण के प्रदर्शन की तुलना करने के लिए कई परीक्षण दृष्टिकोण। इसके अतिरिक्त, हम स्थानिक रूप से स्थानीय सीमा स्थितियों का उपयोग करते हैं। गैर-स्थानीय सीमा मानचित्रों का उच्च-आवृत्ति सन्निकटन उत्पन्न करता है स्थानिक रूप से स्थानीय सीमा की स्थितियाँ। आइए ध्यान दें कि उच्च-फ्रीक्वेंसी बीसी का उपयोग सटीकता को प्रभावित कर सकती है लेकिन उन्हें लागू करना आसान है और कम्प्यूटेशनल जटिलता तुलनात्मक रूप से किफायती है जो उन्हें आकर्षक बनाती है।

स्थानिक विवेकीकरण के लिए, हम एक नई सीमा के साथ लीजेंड्रे-गैलेरकिन वर्णक्रमीय विधि का उपयोग करते हैं अनुकूलित आधार जो यह सुनिश्चित करता है कि परिणामी रैखिक प्रणाली बैंडेड है। एक संगत सीमा-उठाने की प्रक्रिया भी प्रस्तुत की गई है जो खंडों के साथ-साथ कोनों को भी समायोजित करती है। अस्थायी विवेकीकरण को तब एक-कदम समय मार्चिंग योजनाओं का उपयोग करके संबोधित किया जाता है, अर्थात्, पश्चगामी-विभेदन सूत्र क्रम 1 (बीडीएफ1) और समलम्बाकार नियम (टीआर)। ध्यान दें कि इस दृष्टिकोण में गैर-स्थानीय ऑपरेटरों का अनुपयुक्त अस्थायी विवेकीकरण समग्र संख्यात्मक योजना की स्थिरता को नष्ट कर सकता है।

वैकल्पिक रूप से, हम एक दृष्टिकोण प्रदान करते हैं जिसमें प्रारंभिक सीमा-मूल्य समस्या पहले होती है समय-निरंतर गतिशीलता उत्पन्न करने के लिए, सीमा स्थितियों को शामिल करते हुए, अंतरिक्ष में विभेदित किया गया प्रणाली। फिर इस प्रणाली को विभिन्न टाइम-स्टेपिंग विधियों का उपयोग करके समय में विवेकित किया जाता है, जिसके परिणामस्वरूप पूरी तरह से अलग प्रणाली प्राप्त होती है। यह दृष्टिकोण उच्च क्रम की समय-मार्चिंग योजनाओं के उपयोग की सुविधा प्रदान करता है जिसका उपयोग किया जा सकता है संभावित रूप से समग्र प्रणाली की सटीकता में सुधार के लिए। हमारी नवीन पद्धति की प्रभावकारिता को प्रदर्शित करने और अनुभवजन्य रूप से अभिसरण के क्रम की पुष्टि करने के लिए संख्यात्मक परीक्षण प्रस्तुत किए जाते हैं।

Notations

$\mathbb{R}^n$	n-dimensional Eucldian space
$\mathbb{R}_+$	Set of positive real numbers
Ω_i	Interior domain or the computational domain
$\Omega_{l,r}$	Left or right exterior domain, respectively
$\partial\Omega$	Set of boundary of the domain Ω
$\overline{\Omega}$	Closure of a set Ω
supp	Support of a function
$\|\| \cdot \|\|$	Norm
$\mathbb{N}^n$	n-ary Cartesian power of set of non-negative integers (multi-index)
∂_x^α	$\partial^{\alpha_1}/\partial {x_1}^{\alpha_1} \cdots \partial^{\alpha_n}/\partial {x_n}^{\alpha_n}$ where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ a multi-index
∂_t^μ	Riemann-Lioville fractional operator for $\mu \in \mathbb{R}$
$\alpha \cdot \beta$	$\alpha_1\beta_1 + \ldots + \alpha_n\beta_n$ where $\alpha, \beta \in \mathbb{N}^n$ being multi-indices
$\|\alpha\|$	$\alpha_1 + \alpha_2 + \ldots + \alpha_n$ for any multi-index $\alpha \in \mathbb{N}^n$
$\alpha!$	$\alpha_1! \ldots \alpha_n!$ for any multi-index $\alpha \in \mathbb{N}^n$
$\mathsf{C}_0^\infty(\Omega)$	Space of infinitely differentiable, compactly supported functions on Ω
$\mathsf{C}^\infty(\Omega)$	Space of infinitely differentiable functions on Ω
$L^2(\Omega)$	Space of quadratically integrable functions (Lebesgue space) on Ω
$\mathscr{L}, \mathscr{L}^{-1}$	Forward and inverse Laplace transformation, respectively
$\mathscr{F}, \mathscr{F}^{-1}$	Forward and inverse Fourier transformation, respectively
$[L/M]$	Padé approximant of order L, M
$\binom{\alpha}{n}$	Binomial expansion coefficients of $(1+x)^\alpha$

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