

**SEMICLEAN AND FEEBLY CLEAN RINGS  
AND  
RINGS OF CONTINUOUS FUNCTIONS**

**NITIN ARORA**



**DEPARTMENT OF MATHEMATICS  
INDIAN INSTITUTE OF TECHNOLOGY DELHI  
OCTOBER 2014**



**SEMICLEAN AND FEEBLY CLEAN RINGS  
AND  
RINGS OF CONTINUOUS FUNCTIONS**

by

**NITIN ARORA**

Department of Mathematics

Submitted

in fulfilment of the requirements of the degree of  
Doctor of Philosophy

to the



**Indian Institute of Technology Delhi  
October 2014**

# Certificate

This is to certify that the thesis entitled **Semiclean and Feebly Clean Rings and Rings of Continuous Functions** submitted by **Mr. Nitin Arora** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi  
October 2014

**Dr. Subiman Kundu**  
**Professor**  
**Department of Mathematics**  
**Indian Institute of Technology Delhi**

# Acknowledgements

First and foremost I express my sincerest gratitude to my advisor, Professor Subiman Kundu, for his expert guidance and constant encouragement. I especially appreciate the freedom he gave me with my working hours and the faith he showed in me even when results were hard to come. He taught me the basic concepts in algebra and analysis when I was an undergraduate student in the Department of Computer Science at IIT Delhi. I will remain indebted to him for igniting the love for mathematics in me and being an inspiration for pursuing a Ph.D. in Mathematics.

I would like to extend my appreciation to my SRC (Student Research Committee) members as well as to all the faculty members of the Department of Mathematics, IIT Delhi.

I wish to thank my fellow research scholars at the Department of Mathematics, IIT Delhi, especially Varun, Resham, Alok and Vipra.

A Ph.D. in pure mathematics can keep one away from real-life problems. I would like to thank the research group at DIMTS Ltd., especially Mr. Sanjiv Sahai and Dr. Manica Aggarwal for giving me an opportunity to do applied research in solving transportation problems while I freely continued to work on my thesis.

Pursuing a Ph.D. can sometimes be an arduous affair. In such times, it is a blessing to be surrounded by happy, stress-free and dynamic youngsters. I wish to thank all my friends and teachers at the Art of Living and YES!+ for their uplifting

and encouraging presence while I was pursuing research.

Finally, none of this would ever have transpired without the unselfish love and support of my parents, my brother and sister-in-law. Their constant support, patient love and care and unswerving faith has gotten me where I am. This thesis is dedicated to them.

New Delhi

Nitin Arora

October 2014

# Abstract

A ring with unity is said to be *clean* [2; 28] if every element  $x$  can be written as  $x = u + e$ , where  $u$  is a unit and  $e$  is an idempotent, that is  $e^2 = e$ . This notion gains importance since units and idempotents are key elements that determine the structure of a ring. Clean rings are related to several important notions in ring theory, namely, unit regular rings, exchange rings, semiperfect rings and 2-good rings [9; 11; 19; 28; 32]. A ring is said to be *semiclean* if every element is the sum of a unit and a periodic element [38]. A ring is *weakly clean* if every element can be written as either  $u + e$  or  $u - e$ , where  $u$  is a unit and  $e$  an idempotent [1].

Every weakly clean ring is semiclean. We show the existence of semiclean rings that are not weakly clean. Every semiclean ring is 2-clean. The ring  $C(X)$  of all real-valued continuous functions on a Tychonoff space  $X$  is 2-clean, but it is semiclean if and only if  $X$  is strongly zero-dimensional. New classes of semiclean subrings of  $\mathbb{R}$  and  $\mathbb{C}$  are introduced and conditions are given when these rings are clean. We also study when the rings  $C(X, A)$  (respectively,  $C^*(X, A)$ ) of all  $A$ -valued continuous functions (respectively,  $A$ -valued bounded continuous functions) on a Tychonoff space  $X$  are semiclean, especially when  $A$  is a dense clean or semiclean subring of  $\mathbb{R}$  or  $\mathbb{C}$  with 1.

We define a ring  $R$  to be *feebly clean* if every element  $x$  can be written as  $x = u + e_1 - e_2$  where  $u$  is a unit and  $e_1, e_2$  are orthogonal idempotents, that is,

$e_1^2 = e_1$ ,  $e_2^2 = e_2$  and  $e_1e_2 = 0 = e_2e_1$ . Feebly clean rings are a proper generalization of weakly clean rings and the family of semiclean rings properly contains the family of all feebly clean rings. Further properties of feebly clean rings are studied, some of them analogous to those for clean and related rings. Condition is given for  $f \in C(X)$  to be feebly clean and rings of complex-valued continuous functions are studied for feebly clean property.

# Contents

|                                                         |            |
|---------------------------------------------------------|------------|
| <b>Certificate</b>                                      | <b>i</b>   |
| <b>Acknowledgements</b>                                 | <b>iii</b> |
| <b>Abstract</b>                                         | <b>v</b>   |
| <b>List of Symbols</b>                                  | <b>ix</b>  |
| <b>Introduction</b>                                     | <b>1</b>   |
| <b>1 Preliminaries</b>                                  | <b>9</b>   |
| 1.1 Clean Rings and Related Notions . . . . .           | 9          |
| 1.1.1 Properties of Clean and Related Rings . . . . .   | 14         |
| 1.2 Rings of Continuous Functions . . . . .             | 19         |
| 1.2.1 Clean Rings of Continuous Functions . . . . .     | 20         |
| <b>2 Semiclean Rings</b>                                | <b>25</b>  |
| 2.1 Semiclean, Weakly Clean and 2-Clean Rings . . . . . | 26         |
| 2.2 Semiclean Subrings of $\mathbb{R}$ . . . . .        | 32         |
| 2.3 Weakly Clean Subrings of $\mathbb{C}$ . . . . .     | 36         |

---

|          |                                                        |           |
|----------|--------------------------------------------------------|-----------|
| <b>3</b> | <b>Semiclean Rings of Continuous Functions</b>         | <b>39</b> |
| 3.1      | Semicleanliness of $C(X)$ and $C^*(X)$ . . . . .       | 40        |
| 3.2      | $C(X, A)$ for Clean $A$ . . . . .                      | 42        |
| 3.3      | $C(X, A)$ for Semiclean $A$ . . . . .                  | 46        |
| 3.4      | Rings of Complex-valued Continuous Functions . . . . . | 48        |
| <br>     |                                                        |           |
| <b>4</b> | <b>Feebly Clean Rings</b>                              | <b>53</b> |
| 4.1      | Feebly Clean and Related Rings . . . . .               | 54        |
| 4.2      | Properties of Feebly Clean Rings . . . . .             | 59        |
| 4.3      | Rings of Continuous Functions . . . . .                | 65        |
| <br>     |                                                        |           |
|          | <b>Bibliography</b>                                    | <b>75</b> |
| <br>     |                                                        |           |
|          | <b>Index</b>                                           | <b>79</b> |
| <br>     |                                                        |           |
|          | <b>Bio-Data</b>                                        | <b>81</b> |