CONGRUENT NUMBER PROBLEMS, THEIR GENERALIZATION AND SOME RELATED TOPICS

SONAMANI SAHOO

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

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by

SONAMANI SAHOO

Department of Mathematics

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Congruent Number Problems, their generalization, and some related topics"** submitted by **"Ms. Sonamani Sahoo"** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

Dr. R. K. Sharma
Professor
Department of Mathematics
Indian Institute of Technology Delhi
New Delhi, 110016

New Delhi
August 2025

Dr. Ritumoni Sarma
Professor
Department of Mathematics
Indian Institute of Technology Delhi
New Delhi, 110016

New Delhi
August 2025

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Abstract

A congruent number is a positive integer n equal to the area of a right-angle triangle with rational sides. The congruent numbers problem is figuring out if a given positive number is congruent. Determining whether or not a given square-free natural number is congruent is known as the congruent number problem. If the rank of the elliptic curve $E_n : y^2 = x^3 - n^2x$ is positive, then n is a congruent number and conversely. Even though congruent numbers have been examined for centuries, it is still challenging to provide a comprehensive classification.

Several mathematicians have generated infinitely many families of both congruent and non-congruent numbers, satisfying specific properties such as arbitrarily many prime factors. This thesis studied the θ-congruent numbers over quadratic and multi-quadratic number fields and the classification of θ triangles over multi-quadratic number fields. Additionally, we have presented some new families of non $\frac{\pi}{3}$ congruent numbers with some unique properties.

Chapter 1 of the thesis introduces congruent numbers and their generalization. We quickly discuss some prerequisites from number theory, elliptic curves, and introductory algebra in Chapter 2 that we will require later. The complete 2-descent approach, essential to our work, is then described. Monsky's matrices, which we employ later, are described at the end of the second chapter. In Chapter 3, we have presented a criterion for a square-free positive integer that is θ-congruent over multi-quadratic number fields, and we have also outlined some necessary conditions for θ-congruence over quadratic and multi-quadratic number fields.

In Chapter 4, we have studied the classification of θ-triangle over multi-quadratic number fields. In Chapter 5, we constructed new families of non-$\frac{\pi}{3}$ congruent numbers with arbitrarily many prime factors using adapted Monsky Matrices for θ-congruence numbers. We conclude the thesis by outlining our contributions and the direction of future studies in Chapter 6.

कॉन्ग्रुएंट नंबर प्रॉब्लम्स, देयर जनरलाइज़ेशन एंड सम रिलेटेड टॉपिक्स

सारांश

एक कांग्रुएंट नंबर वह धनात्मक पूर्णांक n होता है जो किसी समकोण त्रिभुज का क्षेत्रफल होता है जिसकी भुजाएँ परिमेय होती हैं। कांग्रुएंट नंबर ओं की समस्या यह पता लगाना है कि कोई दी गई धनात्मक संख्या कांग्रुएंट है या नहीं। यह निर्धारित करना कि कोई दी गई वर्ग-मुक्त प्राकृतिक संख्या कांग्रुएंट है या नहीं, कांग्रुएंट नंबर प्रॉब्लम कहलाता है। यदि एलिप्टिक कर्व $E_n: y^2 = x^3 - n^2x$ की रैंक धनात्मक है, तो n एक कांग्रुएंट नंबर है, और इसका उल्टा भी सत्य है। यद्यपि कांग्रुएंट संख्याओं का अध्ययन सदियों से किया जा रहा है, फिर भी उनका एक पूर्ण वर्गीकरण प्रस्तुत करना आज भी एक चुनौतीपूर्ण कार्य है।

कई गणितज्ञों ने विशिष्ट गुणों को संतुष्ट करने वाले, जैसे कि मनमाने रूप से अनेक प्राइम फैक्टर्स, कांग्रुएं और नॉन कांग्रुएं संख्याओं के अनंत फैमिलीज़ उत्पन्न किए हैं। इस शोधप्रबंध में θ- कांग्रुएंट संख्याओं का अध्ययन क्वाड्रेटिक और मल्टी-क्वाड्रेटिक नंबर फील्ड पर किया गया है, साथ ही मल्टी-क्वाड्रेटिक नंबर फील्ड पर θ- त्रिभुज का वर्गीकरण प्रस्तुत किया गया है। इसके अतिरिक्त, कुछ विशेष गुणों वाली नॉन $\frac{\pi}{3}$ कांग्रुएं संख्याओं के कुछ नई फैमिलीज़ भी प्रस्तुत की हैं।

इस शोधप्रबंध के अध्याय 1 में कांग्रुएंट संख्याओं और उनके सामान्यीकरण का परिचय दिया गया है। अध्याय 2 में हम संख्या सिद्धांत, एलिप्टिक कर्व, और प्रारंभिक बीजगणित से कुछ आवश्यक पूर्वज्ञान पर संक्षेप में चर्चा करते हैं, जिसकी आवश्यकता आगे पड़ेगी। हमारे कार्य के लिए आवश्यक कम्प्लीट 2-डिसेंट विधि को इसके बाद वर्णित किया गया है। दूसरे अध्याय के अंत में मौन्स्की मैट्रिक्स का वर्णन किया गया है, जिनका उपयोग हम आगे करेंगे। अध्याय 3 में, हमने मल्टी-क्वाड्रेटिक नंबर फील्ड्स पर किसी स्क्वेयर-फ्री धनात्मक पूर्णांक के θ-कंग्रुएंट होने के लिए एक मापदंड प्रस्तुत किया है, और साथ ही क्वाड्रेटिक तथा मल्टी-क्वाड्रेटिक नंबर फील्ड्स पर θ-कंग्रुएंस के लिए कुछ आवश्यक शर्तों की रूपरेखा भी दी है।

अध्याय 4 में, हमने मल्टी-क्वाड्रेटिक नंबर फील्ड्स पर θ-ट्रायएंगल का वर्गीकरण अध्ययन किया है। अध्याय 5 में, हमने θ-कंग्रुएंस संख्याओं के लिए अनुकूलित मॉन्स्की मैट्रिक्स का उपयोग करते हुए मनमाने रूप से अनेक प्राइम फैक्टर्स वाली नॉन $\frac{\pi}{3}$ कंग्रुएंट संख्याओं की नई फैमिलीज़ निर्मित की हैं।
हम अध्याय 6 में अपने योगदानों की रूपरेखा प्रस्तुत करते हुए और भविष्य के अध्ययन की दिशा को दर्शाते हुए इस शोधप्रबंध का समापन करते हैं।

Contents

List of Symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{Q}^*$	Multiplicative group of nonzero rational numbers
$\mathbb{Q}^{*2}$	Group of squares of elements of $\mathbb{Q}^*$
$\frac{\mathbb{Q}^*}{\mathbb{Q}^{*2}}$	Quotient group of square-free, nonzero rational numbers
$\mathbb{Q}_p$	Field of p-adic numbers
$\mathbb{Z}_n$	Cyclic group of order n
$\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_3} \oplus \cdots \oplus \mathbb{Z}_{n_k}$	Direct sum of cyclic groups
$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \times \cdots \times \mathbb{Z}_{n_k}$	Direct product of cyclic groups
$\mathbb{Z}_{{p_i}^{\gamma_i}}$	The cyclic group with prime-power order
$\mathbb{F}_q$	Finite field with q elements
$ker(\phi)$	Kernel of the homomorphism ϕ
$Img(\phi)$	Image of the homomorphism ϕ
q	a prime power
$a\|b$	a divides b
$a \nmid b$	a does not divide b
$a \equiv b \mod n$	a congruent to b modulo n
$a \equiv b \mod n$	a not congruent to b modulo n

$\forall\ x$	for all$x$
$\|S\|$	number of elements of a finite set$S$
$x \in X$	x belongs to X
$x \notin X$	x does not belong to X
$A \subseteq S$	A is a subset of S
$A \nsubseteq S$	A is not a subset of S
(a, b)	The greatest common divisor of a and b
$\phi(n)$	Euler's phi function of n
$=$	is equal to
$\neq$	is not equal to
$\mathbb{F}_q$	finite field with q elements
$\mathbb{F}_q^*$	the multiplicative group of non zero elements of $\mathbb{F}_q$
$\mathbf{I}_n$	Identity matrix of order n
$\mathbf{O}_n$	Zero matrix of order $n \times n$
$\mathbf{O}$	Zero matrix of any order
$Rank(A)$	Rank of the a matrix A
A^T	Transpose of a matrix A
$det(A)$	Determinant of a square matrix A
$Ker(A)$	Kernel of a matrix A
$Sfp(n)$	Square-free part of a positive integer n