ELLIPTIC EQUATIONS WITH SINGULAR DISCONTINUOUS NONLINEARITIES AND VARIABLE EXPONENTS

by

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To My Family

Certificate

This is to certify that the thesis entitled **“Elliptic equations with singular discontinuous nonlinearities and variable exponents”** submitted by **Ms. Sweta Tiwari** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

In this thesis, we study the existence, non-existence and multiplicity of positive solutions of elliptic equations with discontinuous nonlinearities and elliptic equations with variable exponents in smooth bounded domains.

In the first chapter we give the brief survey and preliminary results used in subsequent chapters.

In the second chapter we prove the existence of the multiple positive solutions of the following elliptic problem with the discontinuous nonlinearity and singular potential on a smooth bounded domain $\Omega \subset \mathbb{R}^2$.

$$-\Delta u \ = \lambda \left(\frac{m(x,u)e^{\alpha u^2}}{|x|^\beta} + u^q \chi_{\{u \le a\}} \right), \ u > 0 \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega,$$

where $0 \le q < 1, 0 < \alpha \le 4\pi$, $\beta \in [0, 2)$ such that $\frac{\beta}{2} + \frac{\alpha}{4\pi} \le 1$ and $m(x, t) \in C^1(\Omega \times \mathbb{R})$ having superlinear growth at infinity. The function $\chi_{\{t \le a\}}$ is defined as $\chi_{\{t \le a\}} = 1$ for $t \le a$ and $\chi_{\{t \le a\}} = 0$ for $t > a$.

In the third chapter we study the singular elliptic problem with the discontinuous nonlinearity on a smooth bounded domain $\Omega \subset \mathbb{R}^2$.

$$-\Delta u \ = \lambda(\chi_{\{u<a\}} u^{-\delta} + h(u)e^{u^2}), \ \ u > 0 \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega,$$

where $a, \lambda > 0$, $0 < \delta < 3$ and $h(u)$ is a smooth nonlinearity that is a "perturbation" of e^{u^2} as $u \to \infty$.

In the fourth chapter we extend the results of the previous chapter to problems in higher dimensions. For $\Omega \subset \mathbb{R}^N (N \geq 3)$, a smooth bounded domain, we discuss the existence of multiple positive solutions of the following elliptic problem with singular and discontinuous nonlinearity:

$$-\Delta u \quad = \lambda \left(u^{2^*-1} + \frac{\chi_{\{u<a\}}}{u^\delta} \right), u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

where $0 < \delta < 3$ and $2^* := 2N/(N-2)$.

In the fifth chapter we study the $W^{1,p(x)}$ versus C^1 local minimizers of functionals related to $p(x)$-Laplacian. For a smooth bounded domain in $\mathbb{R}^N$, we consider the functional $J : W^{1,p(x)}(\Omega) \to I\!R$ defined as

$$J(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} + \int_\Omega \frac{|u|^{p(x)}}{p(x)} - \int_\Omega F(x,u) - \int_{\partial\Omega} G(x,u) d\sigma$$

where $F(x,t) = \int_0^t f(x,s)ds$ and $G(x,t) = \int_0^t g(x,s)ds$. Under the appropriate regularity and growth conditions on the functions $p(x), f(x,t)$ and $g(x,t)$ we prove that a local minimizer of J in C^1 topology is also a local minimizer in $W^{1,p(x)}$ topology.

In the sixth chapter we study the existence and multiplicity results for the following nonlinear Neumann boundary value problem involving the $p(x)-$Laplacian.

$$-\Delta_{p(x)} u + |u|^{p(x)-2} u \quad = \quad f(x,u), \ u > 0 \ \text{ in } \Omega, \ \ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = \lambda u^{q(x)} \ \text{ on } \partial\Omega,$$

where $\Omega \subset I\!R^N, N \geq 2$ is a smooth bounded domain and ν is the outward unit normal to $\partial\Omega$. Under appropriate growth conditions on $f(x,u), p(x)$ and $q(x)$ we show that there exists $\Lambda \in (0, \infty)$ such that the above equation admits two solutions for $\lambda \in (0, \Lambda)$, one solution for $\lambda = \Lambda$ and no solution for $\lambda > \Lambda$.

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