

**DAMPED STOCHASTIC SYSTEM DRIVEN BY
COLORED NOISE : ANALYTICAL AND DIGITAL
SIMULATION STUDIES**

by

CHITRALEKHA MAHANTA

DEPARTMENT OF ELECTRICAL ENGINEERING

Submitted

in fulfilment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

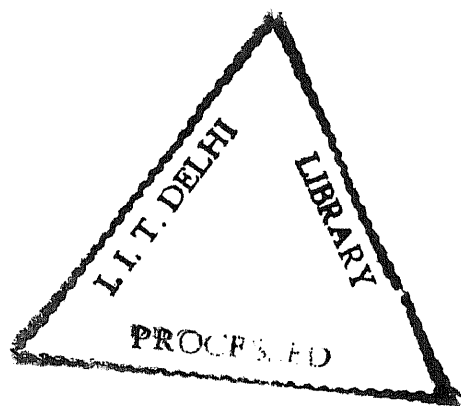
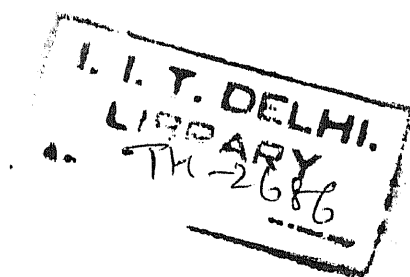
to the



INDIAN INSTITUTE OF TECHNOLOGY , DELHI

JULY, 1999

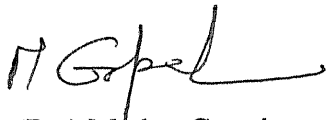
TH
519.21
MAH-D



I dedicate this thesis to
my beloved parents
whose affection is my inspiration

CERTIFICATE

This is to certify that the thesis entitled "Damped stochastic system driven by colored noise: Analytical and digital simulation studies", being submitted by Chitralekha Mahanta to the Department of Electrical Engineering, Indian Institute of Technology, Delhi, for the award of the degree of Doctor of Philosophy, is a bonafide research work carried out by her under our supervision and guidance. The results contained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.



Prof. (Dr.) Madan Gopal
Professor,
Department of Electrical Engineering,
Indian Institute of Technology, Delhi
New Delhi 110016



Dr. T.G. Venkatesh
Assistant Professor,
Department of Electrical Engineering,
Indian Institute of Technology, Madras
Chennai-600036

Acknowledgments

I take this opportunity to express my heartfelt gratitude to my supervisor Prof. M. Gopal. He not only guided and helped me immensely throughout my research work but also kept my morale high during the period of my research. His continuous encouragement was a source of inspiration in my endeavour.

No amount of words can suffice in expressing my sincere thanks to my cosupervisor Dr. T. G. Venkatesh for his constant support, enthusiasm and patience in my research work. His probing comments and insightful suggestions have helped me at every stage of this work. I am indebted to him for an extremely careful and thorough review of the entire thesis.

I thank the laboratory staff of the Multimedia Lab, Deptt. of Electrical Engg., IIT Delhi for extending the excellent facilities of the lab to me for carrying out my research work.

I thank the authority of Indian Institute of Technology, Madras and the staff of the Microprocessor Lab, Deptt. of Electrical Engg., IIT Madras for the hospitality and facilities provided to me during the later part of my research work.

Last but not the least, I thank my friends Anushree, Deepshikha, Monika, Madhavi, Aruna, Jaya, Muna, Shehnaaz and Rajesh for their help and moral support during my research work.

Olaha

Abstract

The thesis concentrates on solving the problem of nonlinear non-Markovian stochastic process associated with a damped stochastic system driven by colored noise. Stochastic nonlinear phenomena are encountered in a number of fields including physics, chemistry, engineering, biology, and humanities. Brownian motion of a particle in a general one-dimensional nonlinear potential driven by a Gaussian Markovian noise acts as a simple mathematical model for a wide variety of systems and phenomena encountered in the above mentioned fields. Such Brownian motions are governed by stochastic differential equations (SDEs) given by, $\ddot{x} = -\gamma\dot{x} - \frac{dU(x)}{dx} + \xi(t)$, where x is the position of the Brownian particle with overdots representing derivatives with respect to time, γ is the damping coefficient, $U(x)$ is the general nonlinear potential and $\xi(t)$ is the colored noise defined by $\langle \xi(t)\xi(s) \rangle = \frac{D}{\tau} \exp(-|t-s|/\tau)$. Here D is the noise strength and τ is the noise correlation time. This thesis is devoted to the study of damped stochastic system driven by colored noise governed by the above mentioned SDE for the stationary probability density function of x .

In this thesis we propose an analytical solution for the problem of damped stochastic system driven by colored noise. First we derive an effective Fokker-Planck equation (EFPE) for the above stochastic process $x(t)$ using the path integral formalism and by applying the steepest descent approximation technique. The stationary probability density function (SPDF) of the above stochastic process is derived analytically using the matrix continued fraction (MCF) method. We then

elaborate our theoretical procedure by taking up a case study on the damped bistable system driven by Gaussian Markovian noise. In order to compare our theoretical results, we carry out extensive digital simulation of the damped bistable system driven by Gaussian Markovian noise. In our digital simulation we compute the SPDF and the mean first passage time (MFPT) of the x process for the cases of white and colored noise. Our theoretical results for the SPDF of damped bistable system driven by colored noise agree very well with our corresponding digital simulation results thus validating the correctness of our theory. Further our digital simulation study is supposed to act as a benchmark for comparing future theoretical results in this area.

Table of contents

	Page No.
1 Introduction	1
1.1 Noise : a General Concept	1
1.2 Motivation to Study Stochastic Nonlinear Dynamical Systems	1
1.3 Statement of the Problem	3
1.4 The Aim of the Thesis	4
1.5 Our Research Work in Nutshell	5
1.6 Organization of the Thesis	6
2 Damped Stochastic System Driven by Colored Noise	8
2.1 Introduction	8
2.2 Mathematical Preliminaries	8
2.2.1 Stochastic Differential Equations	9
2.2.2 The Fokker - Planck Equation	10
2.2.3 The Kramers' Problem	11
2.3 Colored Noise Driven Nonlinear Dynamical Systems	14
2.4 Theories for Overdamped Nonlinear Dynamical Systems Driven by Colored Noise	14
2.4.1 EFPEs	15
2.4.2 The Singular Perturbation Method	17
2.4.3 The Fluctuating Potential Theory (FPT)	17
2.5 The Path Integral Method	18
2.6 Colored Noise Driven Stochastic System with Finite Damping	22
2.7 Example of a Damped Stochastic System : Josephson Junction	23

2.8	The Matrix Continued Fraction Method	23
2.9	Conclusions	26
3	Effective Fokker - Planck Equation : Path Integral Method	27
3.1	Introduction	27
3.2	The Exact Master Equation for the Probability Density Function	28
3.3	Evaluation of the Response Functions	29
3.4	Application of the Path Integral Method	32
3.5	The Steepest Descent Technique	34
3.6	Effective Diffusion Coefficients	35
3.7	The Stationary Probability Density Function	37
3.7.1	Analogy between our EFPE and the FPE for a Damped Stochastic System Driven by White Noise	37
3.7.2	Formula for The SPDF	40
3.8	Conclusions	41
4	The Bistable Potential : a Case Study	42
4.1	Introduction	42
4.2	Physical Significance of the Bistable System	43
4.3	Steps involved in Computing the SPDF	45
4.3.1	The MAP Equation	45
4.3.2	Boundary Conditions for the Equation of the MAP	46
4.3.3	Numerical Computation of the MAP	50
4.3.4	Computation of the Effective Diffusion Coefficients	52
4.3.5	Calculation of the SPDF	52
4.4	Conclusions	53

5 Digital Simulation Studies	55
5.1 Introduction	55
5.2 Multidimensional Langevin Equation	56
5.3 Fox's Second - Order Algorithm	58
5.3.1 Adapted Fox's Algorithm for Digital Simulation	58
5.3.2 The Box - Muller Algorithm	61
5.4 Digital Simulation Procedure	61
5.5 Damped Bistable System Driven by White Noise	63
5.6 Damped Bistable System Driven by Colored Noise	65
5.6.1 Stationary Probability Density Function (SPDF)	66
5.6.2 Mean First Passage Time (MFPT)	71
5.6.3 Failure of the Existing Theories	75
5.7 Conclusions	78
6 Comparison of Our Analytical and Digital Simulation Results	79
6.1 Introduction	79
6.2 SPDF by varying D with fixed γ and τ	80
6.3 SPDF by varying γ with fixed D and τ	85
6.4 SPDF by varying τ with fixed D and γ	90
6.5 Effect of D , γ and τ on the Theoretical Prediction	95
6.6 Conclusions	95
7 Conclusions	97
7.1 Contributions of the Thesis	97
7.2 Scope for future Research Work	99