SYMMETRIES SINGULARITIES AND EXACT SOLUTIONS FOR NONLINEAR SYSTEMS

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TO

MY FATHER, LATE MOTHER

AND

MY SON, DAUGHTER AND WIFE

CERTIFICATE

This is to certify that the thesis entitled:

"SYMMETRIES SINGULARITIES AND EXACT SOLUTIONS FOR NON-LINEAR SYSTEMS", which is being submitted by Mr.Mohsen Hanfy M.Moussa, Research Scholar, Mathematics Department to the Indian Institute of Technology,Delhi, for the award of the DEGREE OF DOCTOR OF PHILOSOPHY in MATHEMATICS, is a record of bonafide research work carried out by him under our guidance and supervision and has fulfilled all the requirements for the submission of this thesis. The results contained in this thesis have not been submitted in part or full, to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

During the last few decades there has been an increased interest in studying problems originating from the realm of nonlinear dynamics. The investigations of this so-called "nonlinear world" has not merely been responsible for revealing a rich and fascinating phenomenology but has, in fact, helped in making more precise some of the concepts and theories developed in the last century mathematics. This has been particularly so for the nonlinear phenomena in physics whose governing differential equations are either nonlinear or involve nonlinear boundary conditions. Specific mention may be made of the Einstein equations of gravitation theory, Yang-Mills equations of elementary particle theory, Navier-Stokes equations or Euler equations of hydrodynamics and so on.

When confronted with nonlinear partial differential equations, which for many a physical and engineering problem require the methods of solution, the standard strategies adopted to date are the following:

i) linearize the given set of equations by invoking certain physical assumptions;

ii) integrate them numerically under appropriate boundary conditions;

iii) reduce them to "integrable" nonlinear equations.

So far as the applications of the first two approaches are concerned, a great deal has been contributed. However, if we examine carefully the work carried out via third approach we find, inspite of it being responsible for leading us to such equations as the Kortweg-de-Vries equation, the Boussinesq equation, the Kadomtsev-Petviashvili equation and many other equations from the Euler equations, this does not seem to have been exploited.

Keeping in view the above remarks and the rich treasure of nonlinear integrable or non integrable equations we have in this thesis carried the application of Lie group analysis alone for obtaining exact solutions of nonlinear integrable equations and in combination with singularity analysis (in particular the Painlevé analysis) for obtaining exact solutions to quite a number of "non integrable" systems. In short, the thesis is devoted to investigating a wide range of applications of continuous symmetry groups to physically important systems of differential equations. In all it comprises seven chapters and is divided into two parts - PART A & PART B. The first chapter collects together those aspects of Lie group theory and other techniques (both group-theoretic and others) which are of importance to the work dealt in Chapters II - VII. More specifically, Part A consisting of Chapters II - III deals with nonlinear partial differential equations of the various physical systems with variable coefficients (viz.

Kadomtsev-Petviashvili equation with variable coefficients, the generalized form of the variable coefficient Kortweg-de-Vries equation, nonlinear Schrödinger equation with variable coefficients and the general form of the coupled reaction diffusion equations in an inhomogeneous medium) via 'Symmetry Approach' due to Steinberg (1979).* Part-B which consists of Chapters IV - VII deals with certain new aspects of the 'Similarity Solutions', 'Travelling Wave Solutions' and 'Painlevé analysis for non integrable equations' for nonlinear differential equations of many a physical system. In particular, Chapter IV deals with the application of a new similarity method to the generalized form of the Klein-Gordon equation that comprises Liouville equation, Double sine-Gordon equation, Dodd-Bullough equation, the generalized form of Kuromoto-Sivashinsky equation and the Boltzmann equation. Chapters V and VI deal, respectively, with the travelling wave solution of a number of single nonlinear differential equations and systems of some coupled nonlinear differential equations. In Chapter VII, the generalized form of two and three dimensional Burger equations, generalized Klein-Gordon equation and Boltzmann equation have been dealt in via Painlevé analysis for integrability and exact solutions.

In the following sections we outline briefly the details of each of the chapters:

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*see Bibliography

CHAPTER I: INTRODUCTION, SURVEY OF LITERATURE AND MATHEMATICAL TOOLS

As mentioned above, this chapter is devoted to the study of various Lie group theoretic techniques and a brief survey and development of the works related to the said techniques. More specifically, we have herein given a brief resumé of the 'Symmetry Approach' due to Steinberg (1979), the new 'Similarity Analysis' of Clarkson and Kruskal (1989), the 'Travelling Wave Solution Technique' put forward by Jeffrey and Xu (1989) for determining the solutions of a number of single or a system of nonlinear partial differential equations of many a physical system. Also included in this chapter are necessary mathematical tools for establishing the integrability of a nonlinear differential equation, namely the 'Painlevé Analysis' as suggested by Weiss, Tabor and Carnevale (1983).

CHAPTER II: ON INVARIANT SOLUTIONS OF THE VARIABLE COEFFICIENTS KADOMTSEV-PETVIASHVILI, GENERALISED KORTWEG-de-VRIES AND NONLINEAR SCHRODINGER EQUATIONS

Herein the variable coefficients Kadomtsev-Petviashvili, generalised Kortweg-de-Vries and nonlinear Schrödinger equations abbreviated respectively as vcKP, vcGKdV and vcNLS, which appear in a wide variety of physical phenomena/applications have been analysed via "Symmetry Method". For the vcKP equation it is shown that the symmetries form an infinite dimensional Lie algebra whereas for vcGKdV and vcNLS equations symmetries constitute Lie algebras of finite dimensions. Further, in all the three cases under investigations, the carry over of the symmetry method has led to certain integrability conditions that are in a form more general than those obtained via the perturbation technique and Painlevé analysis. Also, using the subalgebras of the said Lie algebras, it is shown that in the case of vcKP equation there are three different group theoretic reductions of this equation to Boussinesq equation, vcKdV equation and a solvable form of the second order partial differential equation, depending on certain choices of arbitrary functions of time occurring in the symmetries. For the case of vcGKdV equation the use of subalgebras of the Lie algebras, corresponding to different forms of the arbitrary function of the dependent variable involved in the given equation and that indicates the hierarchy of the KdV equation, helps us to reduce the given equation either to a form that can be solved completely or to a wellknown form like that of Painlevé. An interesting outcome of this study of vcGKdV equation is the deduction of mixed nonlinear KdV equation and its soliton type solution.

Finally, for the vcNLS equation, it is shown that the various subalgebras of the Lie algebras, for different choices of the variable coefficients involved in the given equation,lead to different reductions in the form of coupled nonlinear ordinary differential equations. The search for solutions of these said equations has yielded certain exact solutions which have hitherto been unexplored.

CHAPTER III: ON INVARIANT SOLUTIONS OF A SYSTEM OF GENERALISED COUPLED REACTION-DIFFUSION EQUATIONS

Unlike Chapter II wherein we have taken up the application of the symmetry method to investigate similarity solutions of a number of single partial differential equations*, herein we have carried over its application to a system of generalised coupled nonlinear reaction-diffusion equations in an inhomogeneous medium. In particular, solutions for different forms of the arbitrary functions of the dependent variable involved in the given system have been obtained, using the similarity transformation. Also, indicated is the group theoretic property of the given system and the comparison of the results obtained here with those available in the literature.

*except for vcNLS equation which being a single equation has been treated as a system of two equations.

CHAPTER IV: NEW SIMILARITY SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

In this chapter we present some new similarity solutions of a few wellknown equations from the realms of theoretical physics via a new and direct approach. More specifically, using this direct approach we have been able to reduce the generalised forms of Klein-Gordon and Kuromoto-Sivashnisky equations and the conventional Boltzmann equation to a quadrature whose solutions can be written in terms of Jacobi elliptic functions. Some interesting outcomes of this study are the deduction of the new exact solutions of Liouville equation,Dodd-Bullough equation, sine and double sine-Gordon equations from the solutions of the generalised form of the Klein-Gordon equation. Further, for certain values of the constants involved in the integration of Kuromoto-Sivashinsky equation we have arrived at soliton-type solution that doesn't seem to have been reported in the literature. In order to make this study quite exhaustive we have also carried out independently the search for some new solutions of Liouville equation via the said direct approach. This has yielded two exact solutions one of which seems to have been obtained earlier via Painlevé analysis whereas the other one is completely new.

CHAPTER V: TRAVELLING WAVE SOLUTIONS TO SINGLE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

Keeping in view the importance of solitary wave solutions to nonlinear evolutionary systems we have in this chapter confined our attention to the determination of travelling wave solutions to a number of single partial differential equations governing the physical phenomena in (one, two, three and higher) n-dimensions. To this end we have carried over the procedure due to Jeffrey and Xu (1989) which beside allowing singular solution has provided new solutions to a large number of equations of physical interest. More specifically, under one-dimensional category we have presented the detailed analysis of our search for travelling wave solutions to Kuramoto-Sivashinsky equation and given the list of 18 other equations treated via this procedure in tabular form. Eventhough, all equations tabulated for solutions deserve a mention but notable among them being the following: the fifth order generalised KdV equation, the modified improved Boussinesq equation, the generalised Fisher equation and Sawada-Kotera equation. Under two-dimensional category we have treated in details the Kadomtsev-Petviashvili equation and tabulated the solutions for 5 more equations such as

2-dimensional sine-Gordon equation, Zakharov-Kuzentsov equation etc. For three and higher dimensional equations we have just tabulated the travelling wave and singular solutions of 3- dimensional Boussinesq equation and n-dimensional sine-Gordon equation.

CHAPTER VI: TRAVELLING WAVE SOLUTIONS TO COUPLED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

Herein the procedure due to Jeffrey and Xu for obtaining a class of exact solutions to single nonlinear partial differential equations has been enlarged to obtain exact solutions to coupled nonlinear partial differential equations of physical systems. In particular, beside the singular solutions to the given system, we have obtained travelling wave solutions to classical Boussinesq equation and the Kortweg-de-Vries equation with a self consistent source. The above detailed analysis has eventually helped us to tabulate the solutions for the following important coupled system of nonlinear partial differential equations: Coupled KdV equation, Drinfel'd-Sokolov-Wilson equation and the equations resulting from nonlinear Klein-Gordon equation, Complex Burger equation, Complex modified KdV equation and the nonlinear Schrödinger equation.

CHAPTER VII: ON THE PAINLEVE ANALYSIS OF NON INTEGRABLE PARTIAL DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

The singular manifold expansion developed for obtaining information concerning Lax pairs, Bäcklund transformation etc. for integrable ordinary and partial differential equations is carried to nonintegrable, nonlinear evolution equations such as the higher dimensional Burger equations generalized Klein-Gordon equation and the Boltzmann equation. Unlike the integrable cases of ODE and PDE, the procedure yields, for the non integrable case, a consistency condition which when exploited further leads to certain special solutions. More specifically, in the case of two-dimensional Burger equation the consistency condition obtained is solved via symmetry method and has eventually led to a new solution for it. In the case of three-dimensional Burger equation the consistency condition arrived at is in the form of highly nonlinear PDE of 2nd order in three-dimensions and is not exploited for further solutions. For the generalized form of Klein-Gordon equation the consistency condition turns out to be the same as for the two-dimensional Burger equation except that the roles of the two independent variables are interchanged. In the case of Boltzmann equation the technique has, beside recovering the known solution, yielded a new exact solution.

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