

Lagrange Multipliers in Nonsmooth Optimization Problems

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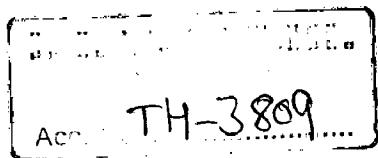
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*Submitted in fulfillment of the requirements
of the degree of Doctor of Philosophy
to the*



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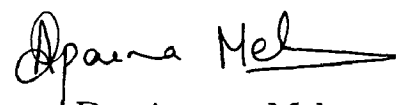
Dedicated To

Puri Sir ...

Certificate

This is to certify that the thesis entitled “**Lagrange Multipliers in Nonsmooth Optimization Problems**” submitted by “**Ms. Anulekha Dhara**” to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.



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Abstract

Any optimization problem may be considered as a minimization problem of a function under certain constraints which can be in the form of inequalities, equations and/or abstract constraints, the latter are constraints that cannot be explicitly expressed in inequality or equation forms. In order to characterize the optimal solution of an optimization problem, the Karush Kuhn Tucker (KKT) optimality conditions or the Lagrange multiplier conditions along with certain convexity or generalized convexity assumptions and under certain mild conditions on the constraint sets called the constraint qualification are studied. The Lagrange multipliers are viewed as the auxiliary variables that are primarily used to derive these optimality conditions for constrained optimization problems. A large effort has been made by the optimization researchers to make the theory more flexible for handling various types of optimization problems encountered in practice. One of the distinguished features of modern optimization theory is to deal with nonsmooth scenario. Nonsmoothness naturally enters into the scene not only through the initial data of the problem but also via various tools applied to study optimization problems.

This thesis is devoted to study the KKT optimality conditions for nonsmooth optimization problems, though in all the chapters except one, the emphasis is on the

minimax programming problems. Also attempts are made to derive the KKT type optimality conditions by relaxing either the convexity or the constraint qualification assumptions or both. By relaxing the convexity assumptions only, the optimality conditions are established via two different approaches, namely the conjugacy approach and the theorem of alternative approach. In the conjugate approach, the optimality conditions are obtained in terms of the conjugate functions and \mathcal{C} -subdifferentials, whereas in the second approach, the same are established in terms of the Clarke subdifferentials. By retaining the convexity structure and relaxing the constraint qualifications, the sequential optimality conditions are derived in terms of the ϵ -subdifferentials. Later in the thesis both convexity and constraint qualifications are dropped to establish approximate optimality conditions. Also the second order optimality conditions for problems involving $C^{1,1}$ data are studied under first and second order constraint qualifications.

Notations

Symbol	Meaning
\emptyset	empty set
\mathbb{N}	set of natural numbers
\mathfrak{R}	real line
$\bar{\mathfrak{R}}; \mathfrak{R}_\infty$	$\mathfrak{R} \cup \{-\infty, +\infty\}$; $\mathfrak{R} \cup \{+\infty\}$, respectively
\mathfrak{R}^n	n -dimensional Euclidean space
$\mathfrak{R}_+; \mathfrak{R}_+^n$	non negative orthant of \mathfrak{R} and \mathfrak{R}^n , respectively
$\mathfrak{R}^Y; \mathfrak{R}_+^Y$	$\prod_Y \mathfrak{R}$; $\prod_Y \mathfrak{R}_+$, respectively
X^*	topological dual space of the space X
$\mathcal{L}(X, Y)$	space of continuous linear mappings from X to Y
$\mathcal{L}^2(X \times X, Y)$	space of continuous bilinear mappings from $X \times X$ to Y
$C(Y)$	space of continuous scalar-valued functions defined on the compact metric space Y
$C_+(Y)$	space of positive functionals on the compact metric space Y

$C^{1,1}$	space of continuously differentiable functions with locally Lipschitz derivatives
$\mathfrak{B}; \bar{\mathfrak{B}}$	open and closed unit balls in an appropriate space, respectively
$x_0 + \delta\mathfrak{B}$	open ball with center at x_0 and radius δ
$cl F$	closure of F
$co F; cl co F$	convex hull and closed convex hull of F , respectively
$aff F; lin F$	affine hull and linear hull of F , respectively
$int F; ri F$	interior and relative interior of F , respectively
$cone F; cone_+ F$	cone and positive cone generated by F , respectively
$F^+; F^\circ$	positive polar cone (or dual cone) and polar cone of F , respectively
$\Lambda_Y; \Lambda_Y^1$	dual cone and normalized dual cone of \mathfrak{R}_+^Y in product topology, respectively
$\phi(F)$	image space of F under ϕ
$dom \phi$	effective domain of $\phi : X \rightarrow \bar{\mathfrak{R}}$
$epi \phi$	epigraph of ϕ
$lev_{\leq \alpha} \phi$	α -lower level set of ϕ
δ_F	indicator function to F
d_F	distance function to F
$proj_F(x_0)$	projection of x_0 to F
$\sigma(\cdot; F)$	support function to F
$\Omega(F)$	effective domain of $\sigma(\cdot; F)$
ϕ^*	conjugate function of ϕ

ϕ_F^*	conjugate function of $(\phi + \delta_F)$
\mathfrak{C}	non empty family of scalar-valued functions
$\phi_{\mathfrak{C}}^*$	\mathfrak{C} -conjugate function of ϕ
$\nabla\phi(x_0)$	Fréchet derivative of ϕ at x_0
$\nabla\phi(x_0)^*$	adjoint operator of $\nabla\phi(x_0)$
$\phi^\circ(x_0; d)$	Clarke directional derivative of ϕ at x_0 in the direction d
$\phi^{\circ\circ}(x_0; d_1, d_2)$	second order Clarke directional derivative of ϕ at x_0 in the direction (d_1, d_2)
$\partial_{\mathfrak{C}}\phi(x_0); \partial_{(\epsilon, \mathfrak{C})}\phi(x_0)$	\mathfrak{C} -subdifferential and (ϵ, \mathfrak{C}) -subdifferential of ϕ at x_0 , respectively
$\partial\phi(x_0); \partial_\epsilon\phi(x_0)$	convex subdifferential and ϵ -subdifferential of ϕ at x_0 , respectively
$\partial_C\phi(x_0)$	Clarke subdifferential or generalized gradient of ϕ at x_0
$J\phi(x_0)$	generalized Jacobian of ϕ at x_0
$\partial_P\phi(x_0); \partial_L\phi(x_0)$	proximal and limiting subdifferential of ϕ at x_0 , respectively
$\partial^P\phi(x_0)$	proximal superdifferential of ϕ at x_0
$\partial_C^2\phi(x_0)$	generalized Clarke Hessian of ϕ at x_0
$\partial_a^2\phi(x_0)$	approximate Hessian of ϕ at x_0
$T(x_0; F)$	tangent cone to F at x_0
$T_C(x_0; F)$	Clarke tangent cone to F at x_0
$T_1(x_0; F)$	first order (inner) tangent set to F at x_0
$T_2(x_0, d_1; F)$	second order (inner) tangent set to F at x_0 in the direction d_1

$O_1(x_0; F)$	first order outer tangent set to F at x_0
$O_2(x_0, d_1; F)$	second order outer tangent set to F at x_0 in the direction d_1
$N(x_0; F); N_\epsilon(x_0; F)$	normal cone and ϵ -normal set to F at x_0 , respectively
$N_C(x_0; F)$	Clarke normal cone to F at x_0
$N_P(x_0; F); N_L(x_0; F)$	proximal and limiting normal cone to F at x_0 , respectively

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