Lagrange Multipliers in Nonsmooth Optimization Problems

By

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Submitted in fulfillment of the requirements

of the degree of Doctor of Philosophy

to the

Indian Institute of Technology Delhi
March 2009

Dedicated To

Puri Sir ...

Certificate

This is to certify that the thesis entitled **"Lagrange Multipliers in Nonsmooth Optimization Problems"** submitted by **"Ms. Anulekha Dhara"** to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Acknowledgements

First and foremost, I owe this thesis to Late Professor M. C. Puri without whose concern and encouragement it would not have been possible for me to meet such a caring supervisor Dr. Aparna Mehra. It was due to him that I met people in my research area who have supported and motivated me throughout my Ph. D. I shall be forever indebted to him for helping me put things in perspective, professionally and personally. His enthusiasm for work, rigor and foresight will always be a source of inspiration and motivation for me.

It gives me immense pleasure to express my deepest gratitude to my supervisor Dr. Aparna Mehra, for her guidance, inspiration and above all her support for carrying out this work. As words are unable to convey even a fraction of feelings, I admit that I can never thank her enough for all she did and her everlasting affection.

I am thankful to IIT Delhi authorities for providing me the necessary facilities for the smooth completion of my work. I would like to give special thanks to my SRC (Student Research Committee) members: Dr. B. S. Panda, Prof. S. G. Deshmukh and Prof. Sanjeev Jain and my external member, Prof. S. K. Neogy, ISI Delhi, for their valuable suggestions. I express my gratitude to all the faculty members and staff of the Department of Mathematics, IIT Delhi, for their encouragement. I take this opportunity to extend my very sincere thanks to Prof. Suresh Chandra

for his advice and constant encouragement. Also I would like to thank the CSIR for providing me financial assistance by awarding me research fellowship throughout my Ph. D. research work.

I would like to convey my sincere thanks to Dr. Joydeep Dutta, IIT Kanpur, Dr. C. S. Lalitha and Dr. Pankaj Gupta, University of Delhi, Delhi for their affectionate support and motivation. I am thankful to the members of Mathematical Programming Group, University of Delhi, for the interactions and discussions which I had with them.

I would also like to thank the reviewers of our papers and the academicians whom I have met over the past years at various conferences and workshops for their valuable suggestions and comments which motivated us to look into areas which we never thought of venturing into.

It appears almost impossible to thank each one of my friends here individually, but I would like them to know that the happy memories of my friendship with them always remain in the deepest corner of my heart. I would particularly like to thank Dr. Tanvi Jain for her generosity and unending encouragement and support. I don't have words to thank my colleague Deepali for providing me a comfortable environment and her help throughout my Ph. D. I would also like to thank Dr. Vaneeta Jindal for her understanding and support.

Last but not the least, I would like to express my deepest and sincerest regards, thanks and gratitude to my parents for their love, encouragement, understanding, unfailing faith, and lot more.

March 2009 Anulekha Dhara

Abstract

Any optimization problem may be considered as a minimization problem of a function under certain constraints which can be in the form of inequalities, equations and/or abstract constraints, the latter are constraints that cannot be explicitly expressed in inequality or equation forms. In order to characterize the optimal solution of an optimization problem, the Karush Kuhn Tucker (KKT) optimality conditions or the Lagrange multiplier conditions along with certain convexity or generalized convexity assumptions and under certain mild conditions on the constraint sets called the constraint qualification are studied. The Lagrange multipliers are viewed as the auxiliary variables that are primarily used to derive these optimality conditions for constrained optimization problems. A large effort has been made by the optimization researchers to make the theory more flexible for handling various types of optimization problems encountered in practice. One of the distinguished features of modern optimization theory is to deal with nonsmooth scenario. Nonsmoothness naturally enters into the scene not only through the initial data of the problem but also via various tools applied to study optimization problems.

This thesis is devoted to study the KKT optimality conditions for nonsmooth optimization problems, though in all the chapters except one, the emphasis is on the minimax programming problems. Also attempts are made to derive the KKT type optimality conditions by relaxing either the convexity or the constraint qualification assumptions or both. By relaxing the convexity assumptions only, the optimality conditions are established via two different approaches, namely the conjugacy approach and the theorem of alternative approach. In the conjugate approach, the optimality conditions are obtained in terms of the conjugate functions and $\mathfrak{C}$-subdifferentials, whereas in the second approach, the same are established in terms of the Clarke subdifferentials. By retaining the convexity structure and relaxing the constraint qualifications, the sequential optimality conditions are derived in terms of the ϵ-subdifferentials. Later in the thesis both convexity and constraint qualifications are dropped to establish approximate optimality conditions. Also the second order optimality conditions for problems involving $C^{1,1}$ data are studied under first and second order constraint qualifications.

Notations

Symbol	**Meaning**
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\Re$	real line
$\bar{\Re}$; $\Re_{\infty}$	$\Re \cup \{-\infty, +\infty\}$; $\Re \cup \{+\infty\}$, respectively
$\Re^n$	n-dimensional Euclidean space
$\Re_+$; $\Re^n_+$	non negative orthant of $\Re$ and $\Re^n$, respectively
$\Re^Y$; $\Re^Y_+$	$\prod_Y \Re$; $\prod_Y \Re_+$, respectively
X^*	topological dual space of the space X
$\mathcal{L}(X, Y)$	space of continuous linear mappings from X to Y
$\mathcal{L}^2(X \times X, Y)$	space of continuous bilinear mappings from $X \times X$ to Y
$C(Y)$	space of continuous scalar-valued functions defined on the compact metric space Y
$C_+(Y)$	space of positive functionals on the compact metric space Y

$C^{1,1}$	space of continuously differentiable functions with locally Lipschitz derivatives
$\mathfrak{B}$; $\bar{\mathfrak{B}}$	open and closed unit balls in an appropriate space, respectively
$x_0 + \delta\mathfrak{B}$	open ball with center at x_0 and radius δ
$cl\ F$	closure of F
$co\ F$; $cl\ co\ F$	convex hull and closed convex hull of F, respectively
$aff\ F$; $lin\ F$	affine hull and linear hull of F, respectively
$int\ F$; $ri\ F$	interior and relative interior of F, respectively
$cone\ F$; $cone_+ F$	cone and positive cone generated by F, respectively
F^+; F°	positive polar cone (or dual cone) and polar cone of F, respectively
Λ_Y; Λ_Y^1	dual cone and normalized dual cone of $\mathfrak{R}_+^Y$ in product topology, respectively
$\phi(F)$	image space of F under ϕ
$dom\ \phi$	effective domain of $\phi : X \to \bar{\mathfrak{R}}$
$epi\ \phi$	epigraph of ϕ
$lev_{\leqq\alpha}\phi$	α-lower level set of ϕ
δ_F	indicator function to F
d_F	distance function to F
$proj_F(x_0)$	projection of x_0 to F
$\sigma(\,.\,; F)$	support function to F
$\Omega(F)$	effective domain of $\sigma(\,.\,; F)$
ϕ^*	conjugate function of ϕ

ϕ_F^*	conjugate function of $(\phi + \delta_F)$
$\mathfrak{C}$	non empty family of scalar-valued functions
$\phi_{\mathfrak{C}}^*$	$\mathfrak{C}$-conjugate function of ϕ
$\nabla\phi(x_0)$	Fréchet derivative of ϕ at x_0
$\nabla\phi(x_0)^*$	adjoint operator of $\nabla\phi(x_0)$
$\phi^\circ(x_0; d)$	Clarke directional derivative of ϕ at x_0 in the direction d
$\phi^{\circ\circ}(x_0; d_1, d_2)$	second order Clarke directional derivative of ϕ at x_0 in the direction (d_1, d_2)
$\partial_{\mathfrak{C}}\phi(x_0)$; $\partial_{(\epsilon,\mathfrak{C})}\phi(x_0)$	$\mathfrak{C}$-subdifferential and $(\epsilon, \mathfrak{C})$-subdifferential of ϕ at x_0, respectively
$\partial\phi(x_0)$; $\partial_\epsilon\phi(x_0)$	convex subdifferential and ϵ-subdifferential of ϕ at x_0, respectively
$\partial_C\phi(x_0)$	Clarke subdifferential or generalized gradient of ϕ at x_0
$J\phi(x_0)$	generalized Jacobian of ϕ at x_0
$\partial_P\phi(x_0)$; $\partial_L\phi(x_0)$	proximal and limiting subdifferential of ϕ at x_0, respectively
$\partial^P\phi(x_0)$	proximal superdifferential of ϕ at x_0
$\partial_C^2\phi(x_0)$	generalized Clarke Hessian of ϕ at x_0
$\partial_a^2\phi(x_0)$	approximate Hessian of ϕ at x_0
$T(x_0; F)$	tangent cone to F at x_0
$T_C(x_0; F)$	Clarke tangent cone to F at x_0
$T_1(x_0; F)$	first order (inner) tangent set to F at x_0
$T_2(x_0, d_1; F)$	second order (inner) tangent set to F at x_0 in the direction d_1

$O_1(x_0; F)$	first order outer tangent set to F at x_0
$O_2(x_0, d_1; F)$	second order outer tangent set to F at x_0 in the direction d_1
$N(x_0; F)$; $N_\epsilon(x_0; F)$	normal cone and ϵ-normal set to F at x_0, respectively
$N_C(x_0; F)$	Clarke normal cone to F at x_0
$N_P(x_0; F)$; $N_L(x_0; F)$	proximal and limiting normal cone to F at x_0, respectively

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