SOME RESULTS ON THE DISPERSION SPECTRUM AND ON MAXIMAL DENSITY OF M-SETS

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Certificate

This is to certify that the thesis entitled:

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which is being submitted by **Soma Gupta** for the award of **Doctor of Philosophy** in Mathematics to the Indian Institute of Technology, Delhi, is a record of bonafide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results obtained in the thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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Abstract

In the thesis, we have looked at problems related to the denseness of different sequences. For any positive integer N, the dispersion $d_N(X)$ of a sequence of points $\{x_n\}$ in X is defined as $d_N(X) = \sup_{x \in X} \min_{1 \leq n \leq N} d(x, x_n)$. Since a sequence $\{x_n\}$ is dense in X if and only if $\lim_{N \to \infty} d_N(x) = 0$, the dispersion is seen as a quantity measuring the denseness of $\{x_n\}$. For irrational α, the sequence $\{n\alpha mod 1\}$ is dense in $[0, 1]$ and $D(\alpha)$ is defined as $D(\alpha) = \limsup_{N \to \infty} N d_N$, where $d_N(X) = \sup_{x \in [0,1]} \min_{1 \leq n \leq N} | x - x_n |$ and $x_n = \{n\alpha \bmod 1\}$. The set of values taken by $D(\alpha)$ is called the *Dispersion Spectrum* in analogy to the well known Markoff Spectrum. Niederreiter[Nie81] studied the concept extensively and stated a few related problems. The problem whether $M(\alpha) \geq M(\beta)$ implies that $D(\alpha) \geq D(\beta)$ was shown to be false by Drobot[Dro86] with the help of a counterexample, which in turn implied that the gaps in the Markoff Spectrum do not necessarily carry over to equivalent gaps in the Dispersion Spectrum. In this work, seven gaps in the Dispersion Spectrum have been identified and also the eight largest gaps in the spectrum above the smallest limit point have been found. Kopetzky and Schnitzer[KS91], besides finding the smallest accumulation point in the Dispersion Spectrum, explicitly obtained every element in the spectrum below this point and so we look for gaps only above this point. A simple proof of the inequality relating Markoff and dispersion constants by Drobot [Dro86] is also given in the thesis. The third problem was finding a real number c above which the spectrum is continuous. We have shown that the Dispersion Spectrum is continuous above $(1 + 2\sqrt{2})/2$.

The other problem studied was of finding the maximal density $\mu(M)$ of sets of integers in which differences given by a set M do not occur. The problem was posed by Motzkin[Mot] in an unpublished problem collection. For a given set of positive integers M, S is an M-set if $a \in S$, $b \in S$ implies $a - b \notin M$, and $\mu(M) = \sup_S \leq \delta(S)$, where $\leq \delta(S) = \limsup_{n \to \infty} S(n)/n$ and $S(n)$ is the number of elements in S less than or equal to n. Cantor and Gordon[CG73] solved the problem for $| M | \leq 2$ and obtained some partial results for the general case. Haralambis[Har77], besides giving some general estimates, determined $\mu(M)$ for most members of the families $\{ 1, j, k \}$ and $\{ 1, 2, j, k \}$. A corrected version of a proof by Haralambis[Har77] of a result on the lower bounds for $\mu(M)$ in terms of a bound for $d(M)$ is given, besides finding $\mu(M)$ for certain infinite families. For the general three element set $M = \{i, j, k\}$ with $\gcd(i, j) = d$, $\mu(M)$ has been found for most families when $i/d \equiv j/d \pmod 2$ and a lower bound for $\mu(M)$ has been found for $i/d \not\equiv j/d \pmod 2$. This bound is very likely the value of $\mu(M)$. $\mu(M)$ has also been found for the set M with elements in arithmetic progression. A lower bound is given for the set $M = \{i, j, i + j\}$ and $\mu(M)$ for certain infinite families of four element set M has also been found.

References

[CG73] D.G. Cantor and B. Gordon. Sequences of Integers with Missing Differences. *Journal of Combinatorial Theory (Series A)*, 14:281–287, 1973.

[Dro86] V. Drobot. On Dispersion and Markov Constants. *Acta Mathematica Hungarica*, 47(1-2):89–93, 1986.

[Har77] N.M. Haralambis. Sets of Integers with Missing Differences. *Journal of Combinatorial Theory (Series A)*, 23:22–33, 1977.

[KS91] H.G. Kopetzky and F.J. Schnitzer. On the Dispersion Spectrum. *Monatshefte für Mathematik*, 112:115–124, 1991.

[Mot] T.S. Motzkin. Unpublished problem collection.

[Nie81] H. Niederreiter. On a Measure of Denseness for Sequences. *Colloquium on Number Theory*, 2:1163–1208, 1981.

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