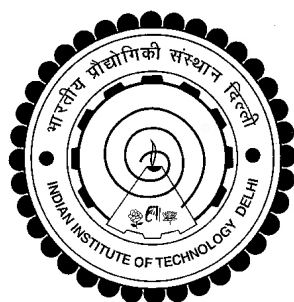


**COMPACT FILTER REGULARIZATION AND ADAPTIVE
SPECTRAL GRAPH WAVELET METHOD WITH
REGULARIZATION FOR PDEs ON GRAPH**

ANKITA SHUKLA



**DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
SEPTEMBER 2020**

**COMPACT FILTER REGULARIZATION AND ADAPTIVE
SPECTRAL GRAPH WAVELET METHOD WITH
REGULARIZATION FOR PDEs ON GRAPH**

by

ANKITA SHUKLA

Department of Mathematics

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



**INDIAN INSTITUTE OF TECHNOLOGY DELHI
SEPTEMBER 2020**

**Dedicated to
My Family**

Certificate

This is to certify that the thesis entitled “**Compact filter regularization and adaptive spectral graph wavelet method with regularization for PDEs on graph**” submitted by “**Ms. Ankita Shukla**” to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.

New Delhi
September 2020

Dr. Mani Mehra
Associate Professor
Department of Mathematics
Indian Institute of Technology Delhi

Acknowledgements

This thesis marks the end of the beautiful journey to achieve my Ph.D. degree. Throughout this journey, I have been supported and guided by several people. I would like to take this opportunity to express my gratitude to all those people.

My first and sincere appreciation goes to Dr. Mani Mehra, my supervisor for all I have learned from her and for her continuous help and support in all stages of this thesis. I would also like to thank her for being an open person to ideas, and for encouraging and helping me to shape my interests and ideas. This thesis would not happen to be possible without the ardent support and care she provided me academically and personally.

It is with immense gratitude that I acknowledge Prof. Günter Leugering, vice president, FAU, Germany. His advice and discussions were invaluable to me. His energetic attitude toward research always inspires me to work harder at my studies. I really appreciate him for always being so supportive.

It gives me great pleasure in acknowledging the help from Dr. S. Sampath. He has been a great source of knowledge and inspiration to me. I want to say thank him for all the advice and discussions during my Ph.D. journey.

I am thankful to IIT Delhi authorities for providing me the necessary facilities for the smooth completion of my Ph.D. I would like to give special thanks to my Student Research Committee members: Dr. S. Sampath, Dr. Harish Kumar and Dr. Anup Singh for their valuable time and suggestions. A special thanks to Prof. K. Sreenadh, Head of the Department and Prof. S. Dharmaraja, former Head of the Department for their support. I express my gratitude to all the faculty members and staff of the Department of Mathematics, IIT Delhi, for their support. I would also like to thank anonymous reviewers and editors of our papers for their useful and enriching suggestions.

My greatest appreciation goes to my senior Dr. Kuldip Singh Patel and my closest friends Dr. Tuhina Mukherjee, Suchismita Patra and Asmita, who was always a great support in all my struggles and frustrations. They have made my time at IIT Delhi much more enjoyable and prosperous. I share the credit of my work with my friends Shaily Verma, Nitu Sharma, Pooja Bansal, Abhilash Sahu, Srashiti Dwivedi, Aman Rani and Vaibhav Mehandiratta. I cannot write the name of each of my friend, but I would like to thank all my friends, I feel blessed to have all of you in my life.

Above all, I would like to thank my parents for their love, blessings, support, encouragement, sacrifice and unwavering belief in me. Without them, I would not be the person I am today. I would like to express my respect and gratitude to my father-in-law for his blessing and unconditional love. I would like to say special thanks to my loving husband, Shailesh, who has been very patient and understanding at every stage of my Ph.D. journey. Thank you for always being there with me. Finally, I want to express my gratitude to Vivek, Neha, Vishal and Mudit for their love and support.

September 2020

Ankita Shukla

Abstract

Inverse problems has numerous important applications in science and industry, which is responsible for its growth in the last few decades. We have proposed high order compact filter as a regularization method for solving some inverse problems which are ill-posed. In this method, we do not have to go back and forth each time between physical and Fourier domain, unlike other filtering methods. We have considered the inverse problem in the field of heat transfer, which is classified into three types: determination of the initial state of the system, determination of the boundary conditions, and identification of the physical parameters or source term identification. We discuss the numerical implementation of the compact filter regularization method to obtain a stable solution for inverse problems one of each type. The problems include homogeneous and non-homogeneous backward heat conduction problems, sideways heat equation, and source term identification problem. The advantage of using high order compact filter regularization method is:

- The proposed method is easy to implement and more efficient as all the computations are done in the physical domain only.
- Compact filters are smooth filter that is preferred as compared to a sharp filter.

We have also provided the error estimates for the exact and regularized solution for these problems. Some numerical experiments are done to validate the efficiency and accuracy of the proposed method. We applied fourth-order compact difference approximation for the second derivative, which proves to have better resolution ability as compared to a standard finite difference approximation.

Nonetheless, the proposed high order compact filter regularization method is very convenient and works well to stably solve inverse problems on flat geometries, we do not know how to apply them for inverse problems defined on complex geometries or graph structure. To solve inverse problems on the graph structure, we have proposed spectral graph wavelet (SGW) as a regularization method. The concept of spectral graph wavelet is based on the graph Laplacian and its construction is done in the spectral graph domain.

In order to validate the applicability of the spectral graph wavelet-based method on a graph, we have initially solved some direct problems. A fast adaptive spectral graph wavelet is developed to solve some PDEs on graph (network-like structures). In the proposed method, SGW serves the following purposes:

- The same matrix is used for the numerical approximation of the graph laplacian as well as for the construction of spectral graph wavelet.
- Wavelet coefficients are used as an indicator function to guide the refinement of the node arrangement which leads to the generation of an adaptive grid.

The proposed method is first applied to solve linear parabolic PDEs on two different graphs. The stability analysis of the proposed method is derived which shows that the method is unconditionally stable. To consider a more general and real-world problem, water wave propagation in open channel networks is analyzed. This problem can be mathematically modeled as Burgers' equation (parabolic PDE having non-linear advection term) on a star-shaped connected graph. The numerical illustrations show that the method accurately captures the emergence of the localized patterns at all the scales, including the junction of the network, and the node arrangement is accordingly adapted. The convergence of the given method is verified, and the efficiency is discussed in terms of CPU time.

Finally, we have proposed the spectral graph wavelet regularization for inverse problems defined on the graph structure. In order to have an efficient solution, we have also used SGW for adaptivity. The numerical implementation of the spectral graph wavelet-based method is discussed mainly to solve the backward heat conduction problem on a graph arising from regular one-dimensional and two-dimensional mesh. Moreover, the

error estimate between the exact and wavelet regularized solution is derived. Numerical results show that we can obtain a stable and efficient solution using the adaptive spectral graph wavelet regularization method.

सार

इनवर्स प्रोब्लेम्स का विज्ञान और उद्योग में कई महत्वपूर्ण अनुप्रयोग हैं, जो पिछले कुछ दशकों में इसकी वृद्धि के लिए जिम्मेदार है। हमने कुछ इनवर्स प्रोब्लेम्स को हल करने के लिए एक नियमितीकरण विधि के रूप में उच्च क्रम वाले कॉम्पैक्ट फ़िल्टर का प्रस्ताव किया है जो इल पोस्ट हैं। इस पद्धति में, हमें भौतिक और फूरियर डोमेन के बीच, अन्य फ़िल्टरिंग विधियों के विपरीत, हर बार पीछे नहीं जाना पड़ता है। हमने गर्मी हस्तांतरण के क्षेत्र में व्युत्क्रम समस्या पर विचार किया है, जिसे तीन प्रकारों में वर्गीकृत किया गया है: प्रणाली की प्रारंभिक स्थिति का निर्धारण, सीमा की स्थिति का निर्धारण, और भौतिक मापदंडों या स्रोत शब्द की पहचान की पहचान। हम प्रत्येक प्रकार के इनवर्स प्रोब्लेम्स के लिए एक स्थिर समाधान प्राप्त करने के लिए कॉम्पैक्ट फिल्टर नियमितीकरण विधि के संख्यात्मक कार्यान्वयन पर चर्चा करते हैं। समस्याओं में समरूप और गैर-सजातीय बैकवर्ड हीट कंडक्शन समस्याओं, साइडवेज़ हीट समीकरण और स्रोत शब्द पहचान समस्या शामिल हैं। उच्च क्रम कॉम्पैक्ट फ़िल्टर नियमितीकरण विधि का उपयोग करने का लाभ यह है:

- प्रस्तावित विधि को लागू करना आसान है और अधिक कुशल है क्योंकि सभी गणना केवल भौतिक डोमेन में की जाती हैं।
- सघन फिल्टर्स की तुलना में कॉम्पैक्ट फिल्टर्स स्मूथ फिल्टर होते हैं।

हमने इन समस्याओं के सटीक और नियमित समाधान के लिए त्रुटि अनुमान भी प्रदान किया है। प्रस्तावित विधि की दक्षता और सटीकता को मान्य करने के लिए कुछ संख्यात्मक प्रयोग किए जाते हैं। हमने दूसरे व्युत्पन्न के लिए चौथे क्रम के कॉम्पैक्ट अंतर सन्निकटन को लागू किया, जो एक मानक परिमित अंतर सन्निकटन की तुलना में बेहतर रिज़ॉल्यूशन क्षमता है।

बहरहाल, प्रस्तावित उच्च क्रम कॉम्पैक्ट फ़िल्टर नियमितीकरण विधि बहुत सुविधाजनक है और प्लैट ज्यामितीयों पर इनवर्स प्रोब्लेम्स को हल करने के लिए अच्छी तरह से काम करती है, हम नहीं जानते कि उन्हें जटिल ज्यामितीय या ग्राफ संरचना पर परिभाषित इनवर्स प्रोब्लेम्स के लिए कैसे लागू किया जाए। ग्राफ संरचना पर इनवर्स प्रोब्लेम्स को हल करने के लिए, हमने एक नियमितीकरण विधि के रूप में वर्णक्रमीय ग्राफ वेवलेट (SGW) का प्रस्ताव किया है। वर्णक्रमीय ग्राफ वेवलेट की अवधारणा ग्राफ लाप्लासियन पर आधारित है और इसका निर्माण दर्शक में किया जाता है।

ग्राफ पर वर्णक्रमीय ग्राफ वेवलेट-आधारित पद्धति की प्रयोज्यता को मान्य करने के लिए, हमने शुरू में कुछ प्रत्यक्ष समस्याओं को हल किया है। एक तेज़ अनुकूली वर्णक्रमीय ग्राफ वेवलेट को कुछ PDEs को ग्राफ (नेटवर्क जैसी संरचनाओं) को हल करने के लिए विकसित किया जाता है। प्रस्तावित विधि में, SGW निम्नलिखित उद्देश्यों को पूरा करता है:

- उसी मैट्रिक्स का उपयोग ग्राफ लैपलियन के संख्यात्मक अनुमान के साथ-साथ वर्णक्रमीय ग्राफ वेवलेट के निर्माण के लिए किया जाता है।
- वेवलेट गुणांक को नोड व्यवस्था के शोधन के लिए एक संकेतक फ़ंक्शन के रूप में उपयोग किया जाता है जो एक अनुकूली ग्रिड की पीढ़ी की ओर जाता है।

प्रस्तावित विधि को पहले दो अलग-अलग रेखांकन पर रैखिक परवलयिक PDEs के समाधान के लिए लागू किया जाता है। प्रस्तावित विधि का स्थिरता विश्लेषण व्युत्पन्न है जो दर्शाता है कि विधि बिना शर्त के स्थिर है। अधिक सामान्य और वास्तविक दुनिया की समस्या पर विचार करने के लिए, खुले चैनल नेटवर्क में जल तरंग प्रसार का विश्लेषण किया जाता है। यह समस्या गणितीय रूप से एक स्टार-आकार से जुड़े ग्राफ पर बर्गर के समीकरण (पैराबोलिक PDE गैर-रेखीय संचलन अवधि वाले) के रूप में तैयार की जा सकती है। संख्यात्मक चित्रण से पता चलता है कि विधि नेटवर्क के जंक्शन सहित सभी पैमानों पर स्थानीयकृत पैटर्न के उद्भव को सटीक रूप से पकड़ती है, और नोड व्यवस्था तदनुसार अनुकूलित होती है। दी गई विधि के अभिसरण को सत्यापित किया गया है, और CPU समय के संदर्भ में दक्षता पर चर्चा की गई है।

अंत में, हमने ग्राफ संरचना पर परिभाषित के लिए वर्णक्रमीय ग्राफ तरंगिका नियमितीकरण का प्रस्ताव दिया है। कुशल समाधान करने के लिए, हमने अनुकूलन के लिए SGW का भी उपयोग किया है। वर्णक्रमीय ग्राफ वेवलेट-आधारित पद्धति का संख्यात्मक कार्यान्वयन मुख्य रूप से नियमित एक-आयामी और दो-आयामी जाल से उत्पन्न होने वाले ग्राफ पर बैकवर्ड हीट कंडक्शन समस्या को हल करने के लिए चर्चा की गई है। इसके अलावा, सटीक और तरंगिका नियमित समाधान के बीच त्रुटि का अनुमान है। संख्यात्मक परिणाम बताते हैं कि हम अनुकूली वर्णक्रमीय ग्राफ वेवलेट नियमितीकरण विधि का उपयोग करके एक स्थिर और कुशल समाधान प्राप्त कर सकते हैं।

Contents

Certificate	i
Acknowledgements	iii
Abstract	v
List of Figures	xiii
List of Tables	xix
List of Symbols	xxi
1 Introduction	1
1.1 Inverse Problem	4
1.2 Regularization	7
1.2.1 Some commonly used regularization methods	8
1.2.2 Compact filter regularization	11
1.3 Spectral graph wavelet method for PDEs	12
1.3.1 Spectral graph wavelet	14
1.3.2 Adaptive node arrangement	24
1.3.3 Spectral graph wavelet regularization	27
1.4 Layout of the Thesis	27

2	Two stable methods with fourth-order compact difference scheme for backward heat conduction problem	31
2.1	Problem formulation	34
2.2	Numerical approximation of second derivative	35
2.2.1	Fourth-order finite difference approximation (FDA4)	35
2.2.2	Fourth-order compact difference approximation (CDA4)	36
2.2.3	Fourier analysis of differencing errors	37
2.3	Fourier analysis of the numerical schemes	39
2.3.1	Fourth order finite difference scheme (FDS4)	41
2.3.2	Fourth order compact difference scheme (CDS4)	42
2.4	Regularization method	43
2.4.1	Quasi-reversibility regularization	44
2.4.2	Compact filter regularization	51
2.5	Numerical results	55
2.5.1	Results for non-regularized PDE	55
2.5.2	Results using quasi-reversibility regularization	59
2.5.3	Results using compact filter regularization	61
3	Nonhomogeneous backward heat conduction problem: compact filter regularization and error estimate	65
3.1	Problem formulation	66
3.2	Compact filter regularization	69
3.2.1	Error estimate	69
3.2.2	Numerical results	75
4	Compact filter regularization method for some other inverse problems	81
4.1	Sideways heat equation	83
4.1.1	Error estimates	87
4.1.2	Numerical results	92
4.2	Source term identification problem	98

4.2.1	Error estimate	100
4.2.2	Numerical results	105
5	Fast adaptive spectral graph wavelet method for PDEs on graph	109
5.1	Problem formulation	111
5.2	Linear PDE model	114
5.2.1	ASGWM for linear PDE	115
5.2.2	Numerical results	117
5.3	Burgers' equation model	126
5.3.1	ASGWM for Burgers' equation	128
5.3.2	Numerical results	129
6	Spectral graph wavelet regularization with wavelet adaptivity for inverse problem	139
6.1	Spectral graph wavelet for path and 2-D grid graph	140
6.1.1	Reconstruction and compression error	142
6.1.2	Adaptive node arrangement	145
6.2	Spectral graph wavelet regularization	148
6.2.1	Error estimate	153
6.3	Numerical results	155
7	Conclusion and future directions	163
7.1	Conclusion	163
7.2	Future directions	165
	Bibliography	167
	List of Publications	179
	Bio-Data	181

List of Figures

1.1	Transfer function versus wavenumber for no filtering and fourth-order compact filtering with $\alpha = 0.4, 0.45$ and 0.49	13
1.2	The star graph $\mathcal{G}$ with $s = 3$	15
1.3	The discrete star graph $\mathcal{G}$ with $s = 3$	16
1.4	The scaling kernel $h(\lambda)$ and wavelet generating kernels $g(t_j(\lambda))$ for $J = 4, \lambda_{max} = 4.58$	20
1.5	The scaling and wavelet function for $J = 4, \lambda_{max} = 4.58$	21
1.6	Fast SGST and SWGT.	23
1.7	The wavelet kernel $g(\lambda)$ and its approximate polynomial p_4 using $M_4 = 20$	23
1.8	Active node points in a star-shaped network.	26
2.1	Plot of modified wavenumber versus wavenumber.	39
2.2	Plot of growth rate versus wavenumber for PDE, FDS4 and CDS4 with $h = 0.024, \Delta t = 0.0001$	43
2.3	Plot for comparing the dispersion relation of PDE and CDS4 using different set of h and Δt (CDS4 ₁ : $h = 0.05, \Delta t = 0.001$; CDS4 ₂ : $h = 0.005, \Delta t = 0.0001$).	44
2.4	Plot of growth rate versus wavenumber for MFDS4 and MCDS4 with $h = 0.024, \Delta t = 0.0001$	48

2.5	Plot for comparing the dispersion relations of MPDE, MFDS4 and MCDS4 using different set of h and Δt ($MCDS4_1$: $h = 0.05$, $\Delta t = 0.001$; $MCDS4_2$: $h = 0.005$, $\Delta t = 0.0001$).	49
2.6	Plot of modified wavenumber versus wavenumber.	53
2.7	Plot of growth rate versus wavenumber for CDS4 and CF4.	54
2.8	Plot for exact solution of PDE and numerical solution by CDS4 for example 1 at final time $T = 1.15 \times 10^{-3}$ using noisy data.	56
2.9	Plot of error for example 1 without regularization (a) with noise δ at fixed $T = 10^{-2}$ and (b) with different T at fixed $\delta = 10^{-3}$	57
2.10	Plot for exact solution of PDE and numerical solution by CDS4 for example 2 at final time $T = 2.5 \times 10^{-3}$ using noisy data.	58
2.11	Plot of error for example 2 without regularization (a) with noise δ at fixed $T = 10^{-2}$ (b) with different T at fixed $\delta = 10^{-3}$	58
2.12	Plot for exact solution of PDE and QR regularized solution for example 1 using CDS4 with final time (a) $T = 0.01$, (b) $T = 0.1$	59
2.13	Plot for exact solution of PDE and QR regularized solution for example 2 using CDS4 with final time (a) $T = 0.01$, (b) $T = 0.1$	60
2.14	Plot of error for example 1 with QR regularization (a) with noise δ at fixed $T = 10^{-2}$ (b) with different T at fixed $\delta = 10^{-3}$	61
2.15	Plot of error for example 2 with QR regularization (a) with different noise δ at fixed $T = 10^{-2}$ (b) with different T at fixed $\delta = 10^{-3}$	61
2.16	Plot for exact solution of PDE and CF regularized solution for example 1 using CDS4 at final time (a) $T = 10^{-2}$, (b) $T = 10^{-1}$	63
2.17	Plot of error for example 1 with CF regularization (a) with noise δ at fixed $T = 10^{-2}$ (b) with different T at fixed $\delta = 10^{-3}$	63
2.18	CPU time at fixed $T = 10^{-2}$ for example 1 after CF regularization using FDS4 and CDS4.	64
2.19	Error versus α for example 1.	64

3.1	Plot of exact and CF regularized solution for example 1 with $\delta = 6 \times 10^{-3}$.	77
3.2	Plot of error for example 1 (a) with noise δ at $t = 0.01$ and (b) with time T at fixed $\delta = 10^{-3}$.	78
3.3	Error e_u versus α for example 1.	79
3.4	Plot of exact and CF regularized solution for example 2 with $\delta = 6 \times 10^{-3}$.	79
4.1	Plot of exact and regularized solution with $\delta = 10^{-3}$.	97
4.2	Error e_u versus α for example 1.	98
4.3	Plot of exact and CF regularized source term for example 1 with various noise levels.	107
4.4	Error e_f versus noise level δ .	107
5.1	Non uniform grids for a general network.	115
5.2	Test function 1 and corresponding wavelet coefficients.	118
5.3	Adaptive node arrangement at $\epsilon = 10^{-4}$ and $N(\epsilon) = 992$.	119
5.4	Solutions and corresponding adaptive node arrangement for example 1.	120
5.5	Solution and pointwise error at $t = 0.4$ for example 2.	121
5.6	Error versus $N(\epsilon)$ for example 2.	121
5.7	Plot of Θ versus ϵ and error versus ϵ for example 2.	122
5.8	Schematic sketch of the graph $\mathcal{G}$ with $s = 6$.	123
5.9	Scaling and wavelet function for graph $\mathcal{G}$ with $s = 6$.	123
5.10	Test function 2 and corresponding reconstructed function for different values of ϵ .	124
5.11	Variation of compression error versus ϵ for test function 2.	124
5.12	Initial condition and solution at $t = 0.7$ and corresponding adaptive grid for $J = 4$ and $\epsilon = 10^{-4}$.	125
5.13	Schematic sketch of a star graph $\mathcal{G}$.	126
5.14	Test function 1.	130
5.15	Scaling and wavelet coefficients for test function 1.	130
5.16	Adaptive node for test function 1 at $\epsilon = 10^{-4}$ ($N(\epsilon) = 528$).	130

5.17	Reconstructed function for different values of ϵ	131
5.18	Variation of compression error versus ϵ and $N(\epsilon)$ for test function 1. . . .	131
5.19	Solution and corresponding adaptive node arrangement for example 1. . . .	133
5.20	Pointwise error at $t = 0.5$	133
5.21	(a) Error versus $N(\epsilon)$ and (b) Θ versus ϵ for example 1 at $t = 0.25$	134
5.22	Test function 2.	134
5.23	Scaling coefficient for test function 2.	135
5.24	Wavelet coefficients d_k^j at different values of j for test function 2.	135
5.25	Reconstructed function for different values of ϵ	136
5.26	Variation of compression error versus ϵ and $N(\epsilon)$ for test function 2. . . .	136
5.27	Solution and corresponding adaptive node arrangement for example 2. . . .	137
5.28	(a) Error versus $N(\epsilon)$ and (b) Θ versus ϵ for example 2 at $t = 0.4$	137
6.1	The graph arising from regular one-dimensional mesh and its discretization. . . .	141
6.2	The scaling kernel $h(\lambda)$ and wavelet generating kernels $g(t_j(\lambda))$ for $J = 4, \lambda_{max} = 4.037$	142
6.3	The scaling and wavelet function for $J = 4$	143
6.4	Test function 1 and reconstructed function for (a) $\epsilon = 10^{-1}$ (b) $\epsilon = 10^{-4}$	144
6.5	Compression error versus (a) ϵ and (b) $N(\epsilon)$ for test function 1.	144
6.6	Test function 2 and corresponding reconstructed function for different values of ϵ	145
6.7	Compression error versus (a) ϵ and (b) $N(\epsilon)$ for test function 1.	145
6.8	Wavelet coefficients d_k^j at different value of j for two different test functions. . . .	146
6.9	Adaptive node generation for path graph.	147
6.10	Adaptive node generation for 2-D grid graph.	147
6.11	Functions and the corresponding adaptive node arrangements in one-dimensional setting for $R = 0.1$ and $M = 6$	147
6.12	Function and the corresponding adaptive node arrangement in two-dimentional setting for $R = 0.1$ and $M = 6$	148

6.13	Test function with noise and corresponding regularized data.	156
6.14	Evolution of the solution and dynamically adapted node arrangement for example 1 using $\epsilon = 10^{-3}$, $R = 0.1$, $M = 6$	157
6.15	Plot of error for example 1 (a) with noise δ at fixed $T = 10^{-2}$ (b) with time T at fixed $\delta = 10^{-3}$	158
6.16	Plot of error e_u versus $N(\epsilon)$ for example 1.	158
6.17	Plot of Θ versus ϵ and error versus ϵ for example 1.	159
6.18	Plot of error for example 2 (a) with noise δ at fixed $T = 10^{-2}$ (b) with time T at fixed $\delta = 10^{-3}$	160
6.19	Wavelet regularized solution and corresponding adapted node arrangement for example 2 using $\epsilon = 10^{-3}$, $R = 0.1$, $M = 6$	161

List of Tables

2.1	Relative error for example 1 without regularization with noisy data. . . .	57
2.2	Relative error for example 2 without regularization with noisy data. . . .	58
2.3	Relative error for example 1 and 2 after CF regularization with noisy data at fixed $\delta = 10^{-4}$	60
2.4	Relative error for example 2 with QR regularization using CDS4 and second order central difference scheme at fixed $\delta = 10^{-3}$	62
2.5	Table to compare the CPU time for example 1 after CF regularization using FDS4 and CDS4 with noisy data.	64
3.1	Comparison of CPU time for example 1 between Tuan et al.[114] and proposed CF4.	77
4.1	Relative error for example 1 for regularized solution with noise level $\delta_1 =$ 10^{-3} and $\delta_2 = 10^{-2}$	97
6.1	Comparison of relative error for example 1 between Fu et al.[38] and AS- GWM.	158
6.2	The performance of ASGWM for example 2 with regularization.	160

List of Symbols

Symbol	Meaning
--------	---------

$\mathbb{R}$	Set of <i>real numbers</i>
--------------	----------------------------

$\mathbb{D}$	k -dimensional bounded space domain in $\mathbb{R}^k$
--------------	---

X	Banach space
-----	--------------

$\forall x$	For <i>all</i> x
-------------	--------------------

$x \in X$	x is a <i>member</i> of X
-----------	-------------------------------

$F_\alpha(\omega)$	Transfer function
--------------------	-------------------

$\mathcal{G}$	Connected graph
---------------	-----------------

V	set of vertices
-----	-----------------

E	set of edges
-----	--------------

A	Adjacency matrix
-----	------------------

$\mathcal{L}$	Laplacian operator
---------------	--------------------

g	Wavelet kernel
-----	----------------

h	Scaling kernel
-----	----------------

$\phi(x)$	Scaling function
-----------	------------------

$\psi(x)$	Wavelet function
-----------	------------------

Ω	space-time domain
ν	heat diffusion coefficient
ω''	modified wavenumber
$\mathcal{L}_2$	Set of square integrable function
h	spatial grid size
Δt	time step size
u_f	filtered value of u
D	space discretization matrix
$\ \cdot\ _2$	L_2 -norm
e_u	relative error with discrete l_2 -norm
δ	noise level
Θ	efficiency coefficient