ALGORITHMIC ASPECTS OF DOMINATION AND ITS VARIATIONS

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ALGORITHMIC ASPECTS OF DOMINATION AND ITS VARIATIONS

by

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Algorithmic Aspects of Domination and Its Variations"** submitted by **"Ms. Arti Pandey"** to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
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Abstract

Domination is one of the important and rapidly growing areas of research in graph theory. A set $D \subseteq V$ is called a *dominating set* of $G = (V, E)$ if $|N_G[v] \cap D| \geq 1$ for all $v \in V$. The Minimum Domination problem is to find a dominating set of minimum cardinality of the input graph. Given a graph G, and a positive integer k, the Domination Decision problem is to determine whether G has a dominating set of cardinality at most k. The *domination number* of a graph G, denoted as $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Though the Minimum Domination problem is known to be NP-hard for general graphs and remains NP-hard even for bipartite graphs, this problem admits polynomial time algorithms for various special graph classes. In this thesis, we propose polynomial time algorithms to compute a minimum dominating set of circular-convex bipartite graphs and triad-convex bipartite graphs. It is also known that for a bipartite graph G, the Minimum Domination problem cannot be approximated within $(1-\epsilon)\ln n$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME$(n^{O(\log\log n)})$. We strengthen the above result by proving that even for star-convex bipartite graphs, the Minimum Domination problem cannot be approximated within $(1-\epsilon)\ln n$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME$(n^{O(\log\log n)})$. We also prove that for bounded degree star-convex bipartite graphs, the Minimum Domination problem is linear-time solvable.

We also study the algorithmic aspects of five variations of dominating set, namely

(i) Open neighborhood locating-dominating set, (ii) Outer-connected dominating set, (iii) Total outer-connected dominating set, (iv) Disjunctive Dominating set, and (v) Dominator coloring.

Let $G = (V, E)$ be a graph and let D be a subset of V. Then (a) D is called an *open neighborhood locating-dominating set (OLD-set)* of G if (i) $N_G(v) \cap D \neq \emptyset$ for all $v \in V$, and (ii) $N_G(u) \cap D \neq N_G(v) \cap D$ for every pair of distinct vertices $u, v \in V$, (b) D is called an *outer-connected dominating set* of G if $|N_G[v] \cap D| \geq 1$ for all $v \in V$, and $G[V \setminus D]$ is connected, (c) D is called a *total outer-connected dominating set* of G, if $|N_G(v) \cap D| \geq 1$ for all $v \in V$, and $G[V \setminus D]$ is connected, (d) D is called a *disjunctive dominating set* of G if for every vertex $v \in V \setminus D$, either v is adjacent to a vertex of D or v has at least two vertices in D at distance 2 from it.

In this thesis, we strengthen the existing literature by proving that the above four variations of domination remains NP-hard for various important subclasses of graphs. We propose some polynomial time algorithms to solve these problems for certain graph classes. We propose approximation algorithms for these problems for general graphs and also prove some approximation hardness results.

A *dominator coloring* of a graph G is a proper coloring of the vertices of G such that every vertex dominates all the vertices of at least one color class. The minimum number of colors required for a proper (dominator) coloring of G is called the *chromatic number* (*dominator chromatic number*) of G and is denoted by $\chi(G)$ ($\chi_d(G)$).

In this thesis, we show that every proper interval graph G satisfies $\chi(G)+\gamma(G)-2 \leq \chi_d(G) \leq \chi(G)+\gamma(G)$, and these bounds are sharp. For a block graph G, where one of the end block is of maximum size, we show that $\chi(G)+\gamma(G)-1 \leq \chi_d(G) \leq \chi(G)+\gamma(G)$. We also characterize the block graphs with an end block of maximum size and attaining the lower bound.

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List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	Set of *natural numbers*
$\mathbb{R}$	Set of *real numbers*
$\forall x$	For *all* x
$x \in X$	x is a *member* of X
$A \subseteq X$	A is a *subset* of X
$\lvert X \rvert$	The *cardinality* of a set X
$A \cap B$	The *intersection* of A and B
$A \cup B$	The *union* of A and B
$A \setminus B$	The *set difference* of A and B
$A \times B$	The *Cartesian product* of A and B
$\emptyset$	The *empty* set
$\square$	The *end of a proof*
$\mid$	such that
$E(G)$	The set of *edges* of G
$V(G)$	The set of *vertices* of G
$d_G(v)$	The *degree* of the vertex v in G

Δ or $\Delta(G)$	The *maximum degree* in G
δ or $\delta(G)$	The *minimum degree* in G
$N_G(v)$	*neighborhood* of v in G
$N_G[v]$	*closed neighborhood* of v in G
$N_G^2(v)$	The set of vertices which are at distance 2 from the vertex v in G
$d_G(u,v)$	The *distance* between u and v in G
$G[H]$	The subgraph of G *induced* by the vertices of H
G^c	The *complement* of an undirected graph G
$G_1 \square G_2$	The *cartesian product* of the graphs G_1 and G_2
K_n	The *complete graph* on n vertices
C_n	The *cycle* on n vertices
P_n	The *path* on n vertices
$\gamma(G)$	The *domination number* of G
$\gamma_{old}(G)$	The *open location-domination number* of G
$\widetilde{\gamma}_c(G)$	The *outer-connected domination number* of G
$\gamma_{toc}(G)$	The *total outer-connected domination number* of G
$\gamma_2^d(G)$	The *disjunctive domination number* of G
$\chi(G)$	The *chromatic number* of G
$\chi_d(G)$	The *dominator chromatic number* of G
P	The class of *deterministic* polynomial-time solvable problems
NP	The class of *nondeterministic* polynomial-time solvable problems
APX	The class of *polynomial-time constant approximable* problems