

A THESIS  
ON  
NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS  
USING SPLINE FUNCTION APPROXIMATIONS

By

Tariq Aziz

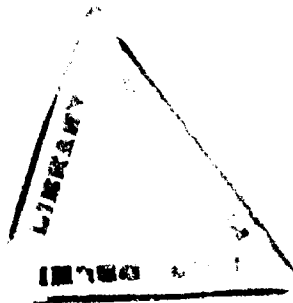
submitted to the Indian Institute of Technology, Delhi  
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in Mathematics

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CERTIFICATE

This is to certify that the thesis entitled, "Numerical Methods for Differential Equations Using Spline Function Approximations", which is being submitted by Mr. Tariq Aziz for the award of the degree, Doctor of Philosophy (Mathematics), to the Indian Institute of Technology, Delhi, is a bonafide record of research work. He has worked for the last three years and nine months under my guidance and supervision.

The thesis has reached the standard, fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

  
( M.K. Jain )

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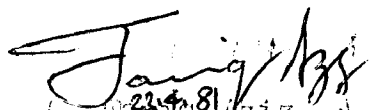
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23.4.81  
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## SYNOPSIS

In recent years, attempts have been made to devise numerical methods for physical problems which lead to one of the following mathematical models:

- (i) Stiff initial value problems
- (ii) Periodic initial value problems
- (iii) Convection-Diffusion problems.

The numerical methods developed may broadly be classified into the following three types: (i) Finite difference methods, (ii) Finite element methods, (iii) Spline function approximation methods.

In this thesis, the spline function approximation methods to obtain numerical solutions of the initial and boundary value problems of ordinary and partial differential equations are investigated. The thesis has been divided into five chapters. This is followed by a bibliography useful for the development and application of the methods discussed in this thesis.

### CHAPTER 1: NUMERICAL METHODS FOR SOLUTION OF DIFFERENTIAL EQUATIONS

The numerical methods, finite difference, finite element and spline function approximation have been discussed for the stiff, periodic and convection-diffusion problems in this

this chapter. We also give the advantage and disadvantages in their applications.

## CHAPTER-2:      SPLINE FUNCTION APPROXIMATIONS

The cubic spline function approximation which has been used both for the approximation of the function and for the solution of ordinary and partial differential equations, is given in detail. The definition of the cubic spline function is extended to the piecewise (spline) functions depending on a parameter  $\omega$ . For  $\omega = 0$ , the spline functions reduce to ordinary cubic spline. The parameter  $\omega$  may also be chosen so as to tailor the spline function to fit the particular problems. Depending on the choice of the parameter, the spline function is termed as, the parametric spline, the adaptive spline, and the spline in tension. The applications of the spline functions are given in the subsequent chapters.

## CHAPTER 3:      NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS USING PARAMETRIC SPLINE FUNCTION

The equivalent linear multistep method is of trigonometric and polynomial order one. The consistency condition is used to determine the parameter  $\omega$ . We obtain

$$\frac{\omega}{2} = \tan \frac{\omega}{2} .$$

This equation has infinite number of roots and the smallest positive non-zero root is given by  $\omega = 8.986818916\dots$ . For

this value of  $\omega$  the spline consistency relations reduce to simple relations which are used to solve initial and boundary value problems of ordinary and partial differential equations. The numerical results are given.

For the zero root, we obtain the cubic spline function.

CHAPTER 4:     NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS  
                   USING ADAPTIVE SPLINE FUNCTION

In this chapter, the equivalent difference methods corresponding to the consistency relations of the adaptive spline functions are listed. It is shown that the difference methods are of exponential and polynomial order one. Further, the difference methods are found to be suitable for convection diffusion problems. The numerical results for the test equations and the Burger equation are presented.

CHAPTER 5:     NUMERICAL METHODS FOR TWO POINT BOUNDARY  
                   VALUE PROBLEMS WITH SIGNIFICANT FIRST  
                   DERIVATIVE USING CUBIC SPLINE

In this chapter, the difference schemes based on cubic spline function have been discussed for the two point boundary value problems with significant first derivative. The difference scheme depends on four arbitrary parameters which may be chosen to obtain schemes of order two, three and four. The difference schemes of order two is unconditionally stable. The scheme of order four gives oscillation free solution with excessive damping when applied to the test convection-conduction model. The numerical results are presented for both linear and nonlinear problems.

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