NEW REPRESENTATIONS OF LIE ALGEBRAS AND IDENTITIES INVOLVING SPECIAL FUNCTIONS

by

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CERTIFICATE

This is to certify that the thesis entitled "NEW REPRESENTATIONS OF LIE ALGEBRAS AND IDENTITIES INVOLVING SPECIAL FUNCTIONS", which is being submitted by Ms. LAKSHMI VARADARAJAN for the award of degree, DOCTOR OF PHILOSOPHY (MATHEMATICS), to the Indian Institute of Technology, Delhi, is a bonafide record of research work done under my guidance and supervision.

The thesis has reached the stage of fulfillment of the requirements and regulations related to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

The theory of Lie groups and Lie algebras plays a significant role in many branches of mathematics and physics. The theory has been fostered, enriched and brought to its elegant and useful form by such great masters as Lie, Engel, Cartan, Killing and Weyl.

For the past fifty years, a deep interest in generating functions was evinced by many mathematicians and many techniques were evolved to obtain them. Among many of such techniques, representation theory of Lie groups occupies a unique position because the theory provides a logical approach and explanation to obtaining generating functions apart from getting them in a very facile manner. Also we obtain recurrence relations and differential equations obeyed by the concerned special functions and a lot of other valid informations. Hence many mathematicians, notably Weisner, Vilenkin, Miller, Kalnins and Manocha, use this method very effectively to many special functions arising in problems of mathematical physics.

In this thesis entitled, 'New representations of Lie algebras and identities involving special functions' the representation theory is used significantly either alone or coupled with Infeld-Hull factorization method and fractional calculus to find out generating functions, recurrence relations, canonical systems of equations etc., for many special functions mainly of the hypergeometric type including basic hypergeometric functions.

Chapter I is introductory in nature. It consists of three parts. The first part consists of a brief resume of Lie theory necessary for the proceeding of our thesis. The second part is on the special functions, their definitions the partial differential equations satisfied by them etc., which occur in this thesis. The third part consists of a survey of fractional calculus and its relevance to us regarding the special functions.

Chapter II evolves a Infeld-Hull type factorization method for a system of linear differintegral operators (second order differential and first order integral). Three factorization types occur in the discussion. Examples of each factorization are given and the eigen functions of the operators are shown to be of hypergeometric type. It is also shown that these factorizations are equivalent to irreducible representations of a class of Lie algebras $\mathcal{G}(a,b)$ and the special functions turn out to be basis vectors of such representations.

In Chapter III we take up factorizations of Chapter II and give models of all the representations of $\mathcal{G}(a,b)$, that is, representations of $\mathcal{J}_3$, $\mathcal{G}(0,1)$ and $sl(2,C)$ by differintegral operators. By using a special operator $\mathcal{D}$ and fractional calculus theory we transform these models into plain differential operator models. This immediately facilitates extensions to local representations of the corresponding Lie groups and induces them on the differintegral models too. By using Weisner's method we obtain various summation identities and generating functions for each of the factorization types.

Chapter IV consists of representation theory involving Jacobi polynomials. We obtain an eight dimensional symmetry algebra from the differential recurrence relations obeyed by the polynomials isomorphic to $sl(3,\mathbb{C})$. We also find out three subalgebras each isomorphic to $sl(2,\mathbb{C})$ and using this fact we extend to Lie group representations and find out many new generating functions and identities in the process.

Chapter V deals with q-Appell functions. In this chapter all the basic Appell functions are taken together and defined in a unified form. In line with the earlier theory by Kalnins, Manocha and Miller for ordinary Appell functions, the recurrence relations are shown and the canonical equations satisfied by each of the q-Appell functions are found and tabulated. Although natural extension to Lie group representation is not possible in these cases, the algebra being infinite dimensional, special cases can be considered by using basic exponential, basic binomial theorem etc. In line with Miller, special cases of extension to group representations for q-Appell functions are dealt with and many generating functions are found.

Chapter VI deals with unitary representations of SU(2). We consider a two variable model of the finite dimensional representation D(2u) of $sl(2,\mathbb{C})$. By using an integral transform we obtain another model of D(2u) in terms of difference-differential operators where the basis functions turn out to be Bessel polynomials. We restrict each of these representations to get models of D_u, which is an irreducible unitary representation of SU(2). This enables us to find out new summation identities in both the cases.

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