ORLICZ FIGÀ-TALAMANCA HERZ ALGEBRAS

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INDIAN INSTITUTE OF TECHNOLOGY DELHI
JUNE 2020

ORLICZ FIGÀ-TALAMANCA HERZ ALGEBRAS

by

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Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

Indian Institute of Technology Delhi
JUNE 2020

To the memory of my grandfather

Karan Singh

Certificate

This is to certify that the thesis entitled **Orlicz Figà-Talamanca Herz algebras** submitted by **Mr. Rattan Lal** to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

JUNE 2020
New Delhi

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Acknowledgements

First and foremost, I would like to express my sincere gratitude to my supervisor, Dr. N. Shravan Kumar for his encouragement and excellent guidance throughout the preparation of this thesis.

I would like to extend my appreciation to my SRC (Student Research Committee) members Prof. V. Ravishankar, Prof. R. Sarma and Prof. S. Sampath as well as to all the faculty members of the Department of Mathematics, IIT Delhi.

My thanks to the staff of the Department of Mathematics, IIT Delhi, for providing me with the necessary facilities to carry out my research work. I also extend my thanks to University Grants Commission, New Delhi, for providing me financial support in the form of research fellowship.

I would like to express my sincere thanks to friends and research scholars at IIT Delhi for their encouragement and support. I would also like to thank my friends Deepak Kaushik and Jatin Wadhwa for constant source of support.

Last, but not least, I would like to dedicate this thesis to my grandparents (Ramphool and Sarjo), my parents (Kamlesh Devi and Mohan Lal), my younger sister (Sunita) and younger brother (Balram) for their love, affection, encouragement and understanding.

New Delhi Rattan Lal

Abstract

In this thesis, we introduce and study the Orlicz Figà-Talamanca Herz algebras denoted as $A_\Phi(G)$, on locally compact groups, which are L^Φ-versions of the classical Figà-Talamanca Herz algebras.

We define the space $A_\Phi(G)$ and show that $A_\Phi(G)$ is a commutative, semisimple and regular Banach algebra with Gelfand spectrum homeomorphic to G. We show that the dual of $A_\Phi(G)$ can be identified isometrically with the space of pseudomeasures $PM_\Psi(G)$. We present some of the functorial properties of the $A_\Phi(G)$ algebras. It is shown that singletons are sets of spectral synthesis. Further, it is also shown that closed subgroups are sets of local spectral synthesis for these algebras. We prove that the set of topological invariant means on the dual of $A_\Phi(G)$ is nonempty and study the cardinality of the set of invariant means.

We characterize the amenability of locally compact groups in terms of the properties of the Orlicz Figà-Talamanca Herz algebras. We establish a characterization of amenable groups in terms of the existence of bounded approximate identities in $A_\Phi(G)$. We prove the equivalence of amenable groups with the weak factorization and cofinite ideals of $A_\Phi(G)$.

We characterize non-degenerate *-representations of $A_\Phi(G)$ and its multiplier algebra $B_\Phi(G)$. We deal with the spectral subspaces associated to a non-degenerate Banach space representation of $A_\Phi(G)$.

We study weakly almost periodic and uniformly continuous functionals on the $A_\Phi(G)$ algebras associated to a locally compact group. We show that a unique invariant mean exists on the space of weakly almost periodic functionals. We give a characterization of discrete groups in terms of the inclusion of the space of uniformly continuous functionals inside the space of weakly almost periodic functionals.

सार

इस शोध में, हम ओरलिक्ज़ फिगाटलामांका हर्ज़ अलजेब्राज का परिचय और अध्ययन लोकली कॉम्पैक्ट ग्रुप्स पर करते है। इसको $A_\Phi(G)$ के रूप में चिह्नित किया जाता है, जो कि क्लासिकल फिगाटलामांका हर्ज़ अलजेब्राज के ओरलिक्ज़ संस्करण है।

हम $A_\Phi(G)$ स्पेस को परिभाषित करते हैं और दिखाते हैं कि $A_\Phi(G)$ एक कम्यूटेटिव, सेमीसिमल और रेगुलर बनाक अलजेब्रा,गेलफां स्पेक्ट्रम ग्रुप G के साथ होमोमोर्फिक है। हम दिखाते है $A_\Phi(G)$ का ड्यूल की पहचान आइसोमेट्रिक रूप से सुडोमेजर स्पेस $PM_\Psi(G)$ के साथ की जा सकती है। हम $A_\Phi(G)$ के कुछ फन्क्टोरिअल प्रॉपर्टीज को प्रस्तुत करते है। यह दिखाया गया है कि सिंगलटन स्पेक्ट्रल सिंथेसिस के सेट है। आगे यह भी दिखाया गया है कि क्लोज्ड सबग्रूप लोकल स्पेक्ट्रल सिंथेसिस के सेट है। हम यह साबित करते है कि $A_\Phi(G)$ के ड्यूल पर टोपोलॉजिकल इनवेरिएंट मीन्स का सेट गैर-रिक्त है और इनवेरिएंट मीन्स के सेट की कार्डिनैलिटी का अध्ययन करें।

हम एमनेबिलिटी ग्रुप्स को ओरलिक्ज़ फिगाटलामांका हर्ज़ अलजेब्राज की प्रॉपर्टीज के संदर्भ में कराक्टेराइज करते है। हम एमनेबिलिटी ग्रुप्स की एक कराक्टेराइजेसन ओरलिक्ज़ फिगाटलामांका हर्ज़ अलजेब्राज में बाउंडिड अप्प्रोक्सिमेट आइडेंटिटी के अस्तित्व के संदर्भ में स्थापित करते है। हम एमनेबिलिटी ग्रुप्स की एक्विवैलेन्स वीक फ़ैक्टरिज़ेशन और कोफाइनेट आइडियल ओरलिक्ज़ फिगाटलामांका हर्ज़ अलजेब्राज में साबित करते है।

हम $A_\Phi(G)$ की नोनडिजनरेट *-रेप्रेसेंटेशन्स और इसकी मल्टीप्लायर अलजेब्रा $B_\Phi(G)$ को कराक्टेराइज करते है। हम $A_\Phi(G)$ की स्पेक्ट्रल सबस्पेसेस नोनडिजनरेट बनाक स्पेस रेप्रेसेंटेशन्स का अध्ययन करते है।

हम वीक्ली ऑलमोस्ट पीरियाडिक और युनिफॉर्मली कंटीन्यूअस फंक्शनलस $A_\Phi(G)$ पर अध्ययन करते है। हम दिखाते हैं कि वीक्ली ऑलमोस्ट पीरियाडिक फंक्शनलस की स्पेस पर एक यूनिक इनवेरिएंट मीन्स मौजूद है। हम डिस्क्रीट ग्रुप्स की कराक्टेराइजेसन युनिफॉर्मली कंटीन्यूअस फंक्शनलस का वीक्ली ऑलमोस्ट पीरियाडिक के अंदर इन्क्लूजन के संदर्भ में प्रस्तुत करते है।

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List of Symbols

Symbol	Meaning
$\forall$	for all
$=$	equal to
$\neq$	not equal to
$\in$	belongs to
$\notin$	does not belong
$\subset$	subset or equal
$\cup, \cap$	union, intersection
$\emptyset$	empty set

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the real line
G	the locally compact Hausdorff group
$C(G)$	the space of complex-valued continuous functions on G
$C_c(G)$	the space of compactly supported continuous functions on G
$C_0(G)$	the space of continuous functions on G vanishing at infinity
$M(G)$	the space of bounded complex Radon measures on G
δ_x	the Dirac-delta measure at x
$\operatorname{supp}(f)$	the support of a complex-valued function f defined on G