ON GROUP RINGS
WITH CERTAIN RESTRICTED CONDITIONS

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ON GROUP RINGS WITH CERTAIN RESTRICTED CONDITIONS

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To Usha, Divyanshi and Kaustubh

Lovely Group Ring of Mine

Certificate

This is to certify that the thesis titled **On Group Rings With Certain Restricted Conditions** submitted by **Mr. Dinesh Udar** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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New Delhi *Dinesh Udar*

Abstract

A ring R is called *restricted right perfect (RRP)* if every proper homomorphic image of R is right perfect. Non-commutative RRP rings were studied by Alberto Facchini and Catia Parolin [11], under the name of *right almost perfect rings*. We have studied RRP group rings. In our study of RRP group rings, necessary conditions for RG to be RRP, but not right perfect have been obtained. A complete characterization of RRP group rings is obtained when $\Delta(G)$ is non-trivial.

In fact it is common in literature that if a ring, R, satisfies the condition that its every proper homomorphic image has a certain property P, then the ring R is called restricted P ring. This has motivated us to introduce and investigate a new class of rings called *restricted boolean rings*. A characterization of commutative restricted boolean rings have been obtained. Our main focus is on the study of restricted boolean group rings which are not boolean. We obtain a complete characterization of non-prime restricted boolean group rings.

A ring R is called *clean* if every element of it is a sum of an idempotent and a unit. A ring R is *neat* if every proper homomorphic image of R is clean. Neat rings were introduced by Warren Wm. McGovern [22]. In our study of neat group rings, we obtain a complete characterization of neat group rings over a field which are not clean. If R is not a field, then necessary conditions are obtained for RG to be neat, but not clean.

We have also studied two subclasses of clean group rings, viz.: strongly P-clean and semiboolean group rings. The class of Strongly P-clean rings is a subclass of semiboolean rings. The class of semiboolean rings lies strictly between the classes of uniquely clean and clean rings. A complete characterization of strongly P-clean group rings has been obtained. Further, semiboolean group rings have also been studied.

We have obtained a generalization of the class of semiboolean rings, called *J-boolean rings*. Various basic properties of these rings are obtained and examples are given to show that the class of J-boolean rings properly contains the classes of uniquely clean, strongly nil clean and semiboolean rings. The J-boolean group rings and skew group rings have been studied. It is investigated whether the results obtained for J-boolean group rings also hold for the skew group rings.

Contents

List of Symbols

Set theory

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$x \in X$	x is a member of X
$x \notin X$	x is not a member of X
$A \subseteq X$	A is a subset of X

Group theory

$< X >$	the subgroup generated by X
$o(g)$	order of an element $g \in G$
C_n	the cyclic group of order n
$C_G(X)$	the centralizer of X in G
$[x, y]$	$= x^{-1}y^{-1}xy$, the commutator of $x, y \in G$
$G' = [G, G]$	the commutator subgroup of G
$\|G : H\|$	index of subgroup H in a group G
gH	right coset of a subgroup H in G
G/H	the quotient group of G by its normal subgroup H
$Z(G)$	the center of G

C_∞	the infinite cyclic group
D_∞	the infinite dihedral group
$\Delta(G)$	$= \{x \in G \| \|G : C_G(x)\| < \infty\}$
$\Delta^+(G)$	$= \{g \in \Delta(G) \| o(g) < \infty\}$
$G \wr H$	wreath product of G by H

Ring theory

$\mathbb{Z}_n$	the ring of integers modulo n
$J(R)$	the Jacobson radical of R
$P(R)$	the prime radical of R
$Z(R)$	the center of R
$\bigoplus_{i\in I} R_i$	direct sum
$\prod_{i\in I} R_i$	direct product
RG	the group ring of G over R
$R *_\theta G$	the skew group ring of G over R
$Aut(R)$	the group of automorphisms of R
$x^{\theta(g)}$	the image of x under the action of $g \in G$
R^G	the fixed subring of G on R
$\mathbb{Z}_{(p)}$	localization of the integers at prime ideal generated by p
$U(R)$	the group of units of the ring R
ωG	the augmentation ideal of RG
$M_n(R)$	full matrix ring of $n \times n$ matrices over R
$T_n(R)$	ring of upper triangular matrices over R
$l(X)$	left annihilator of X
$r(X)$	right annihilator of X
$ann(V)$	the annihilator of an R-module V