OPTIMAL APPROXIMATIONS IN HILBERT SPACES POSSESSING REPRODUCING KERNEL FUNCTIONS

Veena Kaul

Thesis submitted to the Indian Institute of Technology, Delhi
for the award of the Degree of
Doctor of Philosophy

Department of Mathematics
Indian Institute of Technology, Delhi
November 1974

CERTIFICATE

This is to certify that the thesis entitled "Optimal Approximations in Hilbert Spaces Possessing Reproducing Kernel Functions" which is being submitted by Miss Veena Kaul for the award of Doctor of Philosophy in Mathematics to the Indian Institute of Technology, New Delhi, is a record of bonafide research work. She has worked for the last three years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

M.M. Chawla

M.M. Chawla
Assistant Professor
Department of Mathematics
Indian Institute of Technology
New Delhi

ACKNOWLEDGEMENTS

It gives me a great pleasure to express my regards and a profound sense of gratitude to Dr. M.M. Chawla, Assistant Professor, Department of Mathematics, Indian Institute of Technology, New Delhi, for his kind supervision, valuable guidance and constant help throughout the preparation of this thesis. But for his keen interest, this work would not have been possible.

I am particularly grateful to Professor M.K. Jain, Department of Mathematics, I.I.T., Delhi, for his suggestions and encouraging comments in the preparation of my work.

I express my thanks to Professor K.R. Parthasarathy, Head of the Department of Mathematics, Indian Institute of Technology, Delhi, for providing me the necessary facilities in the Department for carrying out my work.

I would also like to record my deep sense of gratitude to my parents, sisters and brother for their patience and affection which have provided impetus to my work.

My thanks are also due to the Council of Scientific and Industrial Research for providing me with a scholarship and to the Indian Institute of Technology, Delhi, for providing facilities during the period this work was done.

Thanks are also due to my colleagues Mr. T.R. Ramakrishnan and Mr. N. Jayrajan for their suggestions and help in the preparation of the thesis. I will be failing in my duty if I do not express my sincere thanks to Dr. S.R. Patel and Mr. V.K. Jain for giving their valuable time in reading the manuscript and many helpful suggestions. My thanks are also due to all my friends who have in one way or the other helped me during the last three years of my research work here.

Finally, I thank Miss Neelam for the excellent typing of the thesis and the utmost care with which she did her job.

Veena Kaul

(Veena Kaul)

Department of Mathematics,
Indian Institute of Technology,
New Delhi-110029.

SYNOPSIS

Classical rules of numerical approximation (say, rules for definite integration) are "best" in the sense that the weights and/or abscissas are determined so that the approximation is exact for all polynomials up to a suitable degree; for example the rules of Chebyshev, Newton-Cotes and Gauss. Various authors have considered the construction of 'optimal' rules by Davis' method by finding the weights and/or abscissas by minimising the norm of the error in a suitable Hilbert space. However, the optimal rules developed so far have been constructed without any requirement of precision for polynomials, and mostly for Hilbert spaces of functions analytic in simply connected regions, mainly circles and certain ellipses. In Chapter I of the thesis, we discuss the construction of optimal linear rules with polynomial precision based on pre-assigned abscissas over Hilbert space possessing a reproducing kernel function. The resulting optimal rules are characterised and specific rules are discussed over two particular Hilbert spaces. We also discuss optimal rules required to be interpolatory for a pre-assigned set of functions of the Hilbert space. In Chapter II, we obtain bounds for the errors of optimal approximations with polynomial precision by making use of the interpolatory property of these optimal rules; the resulting error bounds are of the hypercircle inequality type. Also discussed in this Chapter are some properties of an interpolation operator associated with optimal approximations which are required to be interpolatory for a pre-assigned set of functions of the Hilbert space. In Chapter III, introducing an appropriate Hilbert space, we discuss optimal approximations (with respect to both the abscissas and the weights) for functions analytic in a circular annulus. A family of explicitly determined optimal rules for numerical integration round the unit circle is obtained; and from this, by a suitable transformation, certain families of explicitly determined optimal quadratures for integration over [-1,1] are obtained. In Chapter IV, we obtain explicitly determined optimal rules for the numerical integration of periodic analytic functions by introducing appropriate Hilbert spaces over the basic period-rectangle.

The thesis consists of four Chapters; a brief description of the contents of each Chapter follows:

CHAPTER I: We discuss the construction of optimal linear rules with polynomial precision based on pre-assigned abscissas over a Hilbert space possessing a reproducing kernel function. A rule based on N+1 distinct and pre-assigned points is required to be exact for all polynomials of degree $\leqslant n$, $n \leqslant N$, and the weights are determined by minimizing the norm of the error of the rule. The resulting optimal rules are characterized by the property that they are interpolatory for a certain linear manifold of N-n linearly independent functions of the Hilbert space. For illustration and numerical examples, we consider the Hilbert spaces $H^2(C_r)$ and $L^2(\hat{C}_r)$ of functions analytic in $C_r: |z| = r$, $r > 0$. We first discuss the construction of an optimal rule, with precision for constants, for a point-functional; in this case, the optimal weights are determined explicitly. We then consider some numerically computed optimal rules with polynomial precision for point and integral functionals. The numerical results show that optimal rules with polynomial precision may sometimes be much better (having smaller actual error) then the corresponding optimal rules without any polynomial precision.

CHAPTER II: In Part I of this Chapter, we obtain bounds for the errors of optimal approximations with polynomial precision by making use of the interpolatory property of these optimal rules. The resulting error bounds are of the hypercircle inequality type. Numerical examples are considered over the Hilbert space $L^2(\hat{C}_r)$; the numerical examples show that the error bounds obtained provide quite reliable estimates for the actual errors. In the last part of Chapter I we discussed optimal approximations which were required to be interpolatory for a certain pre-assigned set of functions of the Hilbert space. In Part II of the present Chapter we discuss some properties of an interpolation operator associated with these optimal approximations.

CHAPTER III: Introducing the Hilbert space $H^2(R(r_1,r_2))$, we discuss optimal (with respect to both the abscissas and weights) linear rules for functions analytic in the circular annulus $R(r_1,r_2) = \{z: r_1 < |z| < r_2\}$. We then consider construction of optimal rules for numerical integration round the unit circle

C_1: $|z| = 1$; we obtain a family of explicitly determined optimal rules for $\int_{C_1} f(z)\,|dz|$ over $H^2(R(r_1,r_2))$, $r_1 < 1 < r_2$. The optimal weights are all positive and the optimal nodes are located (equispaced) on $C_{r^*}: |z| = r^*$, $r^* = (r_1 \cdot r_2)^{\frac{1}{2}}$; interestingly, the optimal nodes do not lie on C_1 unless $r_1 r_2 = 1$. Optimal rules over $[-1,1]$ can now be derived from optimal rules over C_1; for if $f(\omega)$ is analytic on $[-1,1]$, then $f(\frac{1}{2}(z+z^{-1}))$ is analytic in $R(r^{-1},r)$ for some $r > 1$. From the family of optimal rules for numerical integration over C_1 we thus obtain certain families of explicitly determined optimal quadratures for integration over $[-1,1]$.

CHAPTER IV: We obtain explicitly determined optimal rules for the numerical evaluation of $\int_{z_0}^{z_0+p} f(z)\,|dz|$, where $f(z)$ is periodic with (complex) period p and analytic on the line ℓ: $z_0 + \sigma p$, $-\infty < \sigma < \infty$. The optimal rules are obtained by minimizing the error norm with respect to both the nodes and the weights over two Hilbert spaces resulting from line and area integral inner products over the basic period rectangle of f. The optimal weights then determined are proportional to the weights of the trapezoidal rule while the optimal nodes are in general located (equispaced) on a certain line segment ℓ'' parallel to ℓ. We also discuss optimal rules with precision for constants; in this case an optimal rule over either of the two Hilbert spaces turns out to be a trapezoidal rule having its nodes lying on the line segment ℓ''.

Some of the results of the thesis have been published in NUMERISCHE MATHEMATIK (1974), and BIT (1973).

CONTENTS

Note: An equation numbered (n) occuring in Chapter m will be referred to as equation (m.n); similarly for theorems, remarks etc.