ONLINE ALGORITHMS FOR GEOMETRIC PROBLEMS

SATYAM SINGH

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

JUNE 2024

ONLINE ALGORITHMS FOR GEOMETRIC PROBLEMS

by

SATYAM SINGH

Department of Mathematics

submitted
in fulfilment of the requirements of the degree of
Doctor of Philosophy

to the

Indian Institute of Technology Delhi

June 2024

Dedicated to
My Family

Certificate

This is to certify that the thesis entitled **ONLINE ALGORITHMS FOR GEOMETRIC PROBLEMS** submitted by **Mr. Satyam Singh** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
June 2024

Prof. Minati De
Department of Mathematics
Indian Institute of Technology Delhi

Acknowledgments

Starting this PhD journey has been truly unforgettable, and I'm grateful for all the help and support I've received from so many people. Due to the huge amount of help I've received, expressing my gratitude feels like an impossible task. But I want to take a moment to thank everyone who has helped me along the way. Your help has been very important to me.

My first note of gratitude goes out to my PhD supervisor, Prof. Minati De, who introduced me to the world of online algorithms. Her guidance, kindness, and contributions have not only helped in my research but also contributed significantly to my professional growth. Working with Prof. De has been an absolute delight, filled with laughter and excitement. Her patience and encouragement during challenging times and exploring new ideas have been invaluable to me. I am highly grateful for her belief in my abilities and for pushing me to reach my full potential. Without Prof. De's guidance and support, completing this thesis would not have been possible. It has been a pleasure to collaborate with Prof. De, and I learned that the best way to get ahead is to be sincere and hard-working.

I extend my gratitude to my Student Research Committee (SRC) members, Prof. Bhawani S. Panda, Prof. Ashutosh Rai, and Prof. Amit Kumar, for their invaluable suggestions and dedicated time. Their insightful questions and suggestions have motivated me to keep progressing in my research. I am also thankful to Prof. Subhas C. Nandy from ISI Kolkata for sparing his valuable time to attend my Council of Scientific and Industrial Research (CSIR) fellowship extension seminar. I would also like to thank my co-authors, whose collaboration and contributions have been integral to the completion of this work. I want to express my heartfelt gratitude to Prof. De's mother, Mrs. Suchitra Devi, for her blessings and delicious homemade food during my visit to their house. Additionally, I am thankful to Mr. Ayan Pal for his warm wishes and blessings during my stay at IIT Delhi.

I also want to thank the system administrators, librarians, and other staff of the Department of Mathematics at the Indian Institute of Technology Delhi from the bottom of my heart. Whenever I needed help, I could not have done it without their support. Furthermore, I am grateful to the Council of Scientific and Industrial Research for providing financial support during my PhD journey. Their support has been crucial in enabling me to pursue my research goals.

I want to take this opportunity to express my heartfelt gratitude to all the wonderful individuals I had the pleasure of meeting during my time at IIT Delhi. Your presence made my PhD journey both productive and enjoyable. Thank you to Abhilash, Akanksha, Ankit, Ankita, Anshu, Anshul, Anuj, Arvish, Astha, Chetan, Diksha, Divyanee, Jaya, Kanupriya, Lalit, Naveen, Navnit, Prerna, Ritika, Ritu, Sarfaraz, Shakti, Shammi, Sridharan, Sujoy, Sushmita, and my lovely juniors Ratnadip, Rumki, Sahiba, Soumen, and Soumyashree for the enjoyable breaks and for creating such a positive and productive atmosphere around me.

I also want to thank Dr Vaibhav Mehandiratta, Dr Abhishek Kumar Singh, Dr Bhawna, Dr Nitin Kumar, Dr Kshitiz Kumar Pandey, Dr Priyamvada, and Dr Abhay Kumar Chaturvedi. The stories you have shared with me during our conversations have truly changed my life and have a huge effect on it. Throughout my PhD journey, you have always been an inspiration to me.

Furthermore, I extend my appreciation to my dear friends Ayush Bhaiya, Biharee Lal, Devyanshi, Divyanshu Bhaiya, Himanshu, Kavin, and Raju for always being supportive even when things were hard. Your support has been very important in giving me the strength and courage to overcome the tough times. A special thanks to Shweta for being a part of my journey and making it special.

There are no words to describe how grateful I am to my family for always loving, encouraging, and helping me with my PhD. Their belief in me has been a constant source of strength and motivation. Thank you so much, my father, Ajai Kumar Singh and mother, Rekha Devi, for all the sacrifices you have made for me. Your constant support and words of encouragement have been the key to my success. I will always be thankful for your love and guidance. Vaishnavi and Vaishali, thank you for always being there for me and understanding me through every challenge. Because you believed in me, I was able to keep going and reach my goals. I am truly *blessed to have such an amazing family, and I dedicate this achievement to every one of you. Thank you for being my constant source of love, support, and inspiration.*

New Delhi Satyam Singh

Abstract

The online model of computation has gained significant interest for more than five decades due to its ability to effectively capture real-world phenomena like the irreversibility of time (decisions) and the unpredictability of the future (input). In this thesis, we study three fundamental graph-theoretic problems: maximum independent set (MIS) problem, minimum dominating set (MDS) problem and its variants, and minimum coloring (MC) problem and two set-theoretic problems: hitting set problem and piercing set problem in the online setup.

The graph-theoretic problems considered here, apart from the minimum connected dominating set (MCDS) problem, are studied in the classical online model. For the online MIS and MDS problems, in the classical online setup, the disastrous $\Omega(n)$ lower bounds on the competitive ratios are known for general graphs over two decades, which also holds for interval graphs, where n is the number of vertices. In this thesis, for the online MIS, MDS and maximum independent dominating set (MIDS) problems, we demonstrate that for a graph with an independent kissing number at most ζ, there exist well-known greedy algorithms that achieve optimal competitive ratios of ζ. In contrast, for the online MCDS problem, we show that there exists a deterministic algorithm having an optimal (asymptotic) competitive ratio of $2(\zeta - 1)$. For the online MC problem, we show that there exists an algorithm that achieves a competitive ratio of at most $\zeta'(\lfloor \log M \rfloor + 1)$, which is an improvement over the existing best-known result of ζ. Here, ζ' and ζ are independent kissing numbers of similarly sized fat objects having widths in the range $[1, 2]$ and $[1, M]$, respectively. The results obtained in this thesis for these problems heavily depend on a graph parameter: independent kissing number. Thus, we investigated the value of the independent kissing number for various families of geometric objects.

Then, we switch our attention to set-theoretic problems in the online setup. It is known that no online algorithm can obtain a competitive ratio better than $\Omega(\log M)$ for hitting M intervals in the range $[1, M]$ using points $\mathbb{Z}$. Due to this pessimistic result, we consider the online hitting set problem for a finite family of translates of an object in $\mathbb{R}^d$ using points in $\mathbb{Z}^d$. For the low dimensional unit balls and hypercubes in $\mathbb{R}^d$ $d \leq 3$), we present deterministic algorithms having almost tight constant competitive ratios. For the unit hypercubes in $\mathbb{R}^d$ ($d \geq 3$), we present an $O(d^2)$-competitive randomized algorithm, while for unit balls in $\mathbb{R}^d$, we present an $O(d^4)$-competitive deterministic algorithm. Additionally, we investigated the lower bounds of the hitting set problem for unit balls in $\mathbb{R}^d$ (for $d \leq 3$) and unit hypercubes in $\mathbb{R}^d$ (for $d \in \mathbb{Z}^+$), and we show that every deterministic algorithm has a competitive ratio of at least $d + 1$. We prove that the competitive ratio of every deterministic online algorithm for piercing intervals in $\mathbb{R}$ is at least $\Omega(n)$. Due to this hopeless lower bound, in this thesis, we consider the piercing set problem for similarly sized fat objects in $\mathbb{R}^d$. We propose a deterministic online algorithm for similarly sized fat objects in $\mathbb{R}^d$. For homothetic hypercubes in $\mathbb{R}^d$ with side length in the range $[1, M]$, we propose a $(3^d\lceil\log_2 M\rceil + 2^d)$-competitive deterministic algorithm. Furthermore, we obtain a deterministic lower bound of the problem for homothetic hypercubes in $\mathbb{R}^d$. Note that piercing translated copies of a convex object is equivalent to the unit covering problem, which is well-studied in the online setup. Our result yields an upper bound of the competitive ratio for the unit covering problem when the corresponding object is any convex object in $\mathbb{R}^d$.

सार

गणना के ऑनलाइन मॉडल ने समय की अपरिवर्तनीयता (निर्णयों) और भविष्य की अप्रत्याशितता (इनपुट) जैसी वास्तविक दुनिया की घटनाओं को प्रभावी ढंग से पकड़ने की अपनी क्षमता के कारण पांच दशकों से अधिक समय से महत्वपूर्ण रुचि प्राप्त की है। इस थीसिस में, हम तीन मौलिक ग्राफ-सैद्धांतिक समस्याओं का अध्ययन करते हैं: अधिकतम स्वतंत्र सेट (एमआईएस) समस्या, न्यूनतम हावी सेट (एमडीएस) समस्या और इसके प्रकार, और न्यूनतम रंग (एमसी) समस्या और दो सेट-सैद्धांतिक समस्याएं: ऑनलाइन सेटअप में हिटिंग सेट समस्या और पियर्सिंग सेट समस्या।

न्यूनतम कनेक्टेड डोमिनेटिंग सेट (एमसीडीएस) समस्या के अलावा, यहाँ पर विचार की गई ग्राफ-सैद्धांतिक समस्याओं का अध्ययन शास्त्रीय ऑनलाइन मॉडल में किया जाता है। ऑनलाइन एमआईएस और एमडीएस समस्याओं के लिए, शास्त्रीय ऑनलाइन सेटअप में, प्रतिस्पर्धी अनुपातों पर विनाशकारी $\Omega(n)$ निचली सीमाएँ दो दशकों से सामान्य ग्राफ़ के लिए जानी जाती हैं, जो अंतराल ग्राफ़ के लिए भी मान्य है, जहाँ n शीर्षों की संख्या है। इस थीसिस में, ऑनलाइन एमआईएस, एमडीएस और अधिकतम स्वतंत्र प्रभुत्व सेट (एमआईडीएस) समस्याओं के लिए, हम प्रदर्शित करते हैं कि अधिकतम ζ पर एक स्वतंत्र न्यूटन संख्या वाले ग्राफ के लिए, ऐसे जाने-माने लालची एल्गोरिदम मौजूद हैं जो ζ के इष्टतम प्रतिस्पर्धी अनुपात को प्राप्त करते हैं। इसके विपरीत, ऑनलाइन एमसीडीएस समस्या के लिए, हम दिखाते हैं कि एक नियतात्मक एल्गोरिदम मौजूद है जिसका इष्टतम (असिमोटोटिक) प्रतिस्पर्धी अनुपात $2(\zeta - 1)$ है। ऑनलाइन एमसी समस्या के लिए, हम दिखाते हैं कि एक एल्गोरिदम मौजूद है जो अधिकतम $\zeta'(\lfloor \log M \rfloor + 1)$ का प्रतिस्पर्धी अनुपात प्राप्त करता है, जो कि ζ के मौजूदा सबसे प्रसिद्ध परिणाम पर एक सुधार है। यहाँ, ζ' और ζ समान आकार की मोटी वस्तुओं की स्वतंत्र न्यूटन संख्याएँ हैं जिनकी चौड़ाई क्रमशः $[1,2]$ और $[1, M]$ की सीमा में है। इन समस्याओं के लिए इस थीसिस में प्राप्त परिणाम एक ग्राफ पैरामीटर पर बहुत अधिक निर्भर करते हैं: स्वतंत्र न्यूटन संख्या। इस प्रकार, हमने ज्यामितीय वस्तुओं के विभिन्न परिवारों के लिए स्वतंत्र न्यूटन संख्या के मूल्य की जांच की।

फिर, हम अपना ध्यान ऑनलाइन सेटअप में सेट-सैद्धांतिक समस्याओं पर केंद्रित करते हैं। यह ज्ञात है कि कोई भी ऑनलाइन एल्गोरिदम $\Omega(\log M)$ से बेहतर प्रतिस्पर्धी अनुपात प्राप्त नहीं कर सकता है, जो कि $[1, M]$ की सीमा में M अंतरालों को हिट करने के लिए बिंदुओं $\mathbb{Z}$ का उपयोग करता है। इस निराशावादी परिणाम के कारण, हम $\mathbb{Z}^d$ में बिंदुओं का उपयोग करके $\mathbb{R}^d$ में किसी ऑब्जेक्ट के ट्रांसलेट के परिमित परिवार के लिए ऑनलाइन हिटिंग सेट समस्या पर

विचार करते हैं। $\mathbb{R}^d$ $(d \leq 3)$ में कम आयामी इकाई गेंदों और हाइपरक्यूब के लिए, हम लगभग सख्त स्थिर प्रतिस्पर्धी अनुपात वाले नियतात्मक एल्गोरिदम प्रस्तुत करते हैं। $\mathbb{R}^d$ $(d \leq 3)$ में इकाई हाइपरक्यूब के लिए, हम एक $O(d^2)$-प्रतिस्पर्धी याहच्छिक एल्गोरिदम प्रस्तुत करते हैं, जबकि $\mathbb{R}^d$ में इकाई गेंदों के लिए, हम एक $O(d^4)$-प्रतिस्पर्धी नियतात्मक एल्गोरिदम प्रस्तुत करते हैं। इसके अतिरिक्त, हमने $\mathbb{R}^d$ $(d \leq 3)$ में इकाई गेंदों और $\mathbb{R}^d$ $(d \in \mathbb{Z}^+)$ में इकाई हाइपरक्यूब्स के लिए हिटिंग सेट समस्या की निचली सीमाओं की जांच की, और हमने दिखाया कि प्रत्येक नियतात्मक एल्गोरिथ्म का प्रतिस्पर्धी अनुपात कम से कम $d+1$ है। हम साबित करते हैं कि $\mathbb{R}$ में पियर्सिंग अंतराल के लिए प्रत्येक नियतात्मक ऑनलाइन एल्गोरिदम का प्रतिस्पर्धी अनुपात कम से कम $\Omega(n)$ है। इस निराशाजनक निचली सीमा के कारण, इस थीसिस में, हम $\mathbb{R}^d$ में समान आकार की मोटी वस्तुओं के लिए पियर्सिंग सेट समस्या पर विचार करते हैं। हम $\mathbb{R}^d$ में समान आकार की मोटी वस्तुओं के लिए एक नियतात्मक ऑनलाइन एल्गोरिदम प्रस्तावित करते हैं। $[1, M]$ की सीमा में साइड लेंथ वाले $\mathbb{R}^d$ में होमोथेटिक हाइपरक्यूब के लिए, हम एक $(3^d\lceil\log_2 M\rceil + 2^d)$-प्रतिस्पर्धी नियतात्मक एल्गोरिदम प्रस्तावित करते हैं। इसके अलावा, हम $\mathbb{R}^d$ में होमोथेटिक हाइपरक्यूब के लिए समस्या की एक नियतात्मक निचली सीमा प्राप्त करते हैं। ध्यान दें कि उत्तल वस्तु की अनुवादित प्रतियों को पियर्स करना यूनिट कवरिंग समस्या के बराबर है, जिसका ऑनलाइन सेटअप में अच्छी तरह से अध्ययन किया गया है। हमारा परिणाम इकाई आवरण समस्या के लिए प्रतिस्पर्धी अनुपात की ऊपरी सीमा प्रदान करता है, जब संगत वस्तु $\mathbb{R}^d$ में कोई उत्तल वस्तु होती है।

Contents

List of Tables

List of Figures