NUMERICAL METHODS FOR
VOLTERRA AND SINGULAR INTEGRAL EQUATIONS

By

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SYNOPSIS

In past, methods for approximating the unknown solution of the second kind Volterra integral equation:

$$y(x) = f(x) + \int_a^x K(x,t,y(t))dt, \quad a \leqslant x \leqslant b, \qquad (1)$$

have been frequently obtained by replacing the integral in each interval by a numerical quadrature formula treating $K(x,t,y(t))$ as a single function. However, a better method for approximating the unknown solution $y(x)$ can be obtained by treating $y(x)$ separately in each interval. Methods obtained for equation (1) using the above concept will have precision and improved accuracy.

In support of the above statement, we may consider the following class of nonlinear second kind Volterra integral equations:

$$y(x) = f(x) + \int_a^x k(x,t)u(y(t))dt, \quad a \leqslant x \leqslant b. \qquad (2)$$

If $k(x,t)$ oscillates more rapidly or is less smooth than the solution $y(x)$, then it may be easily seen that the better approximations for the unknown solution $y(x)$ can be obtained by treating $y(x)$ separately.

In Chapters I and II, we describe some implicit and explicit methods based on the above concept for the approximate solutions of (1). After going into the details of the methods given in Chapters I and II, we may reach the conclusion that any method for (1) which has been obtained by treating $K(x,t,y(t))$ as a single function can be easily modified to give better approximations for the unknown solution $y(x)$ by applying the same approximations on $y(x)$ separately which we have applied on $K(x,t,y(t))$ treating it as a single function. We show that such modifications preserve the strong convergence properties of the methods. The methods described in Chapters I and II are stable, but we shall not go into details of these, keeping in view the space and also our aim is not this.

In Chapter III, we describe a Chebyshev series method for the approximate solutions of the Volterra integral equation

$$by(x) = f(x) + \lambda \int_{-1}^{x} (K(x,t)/(x-t)^{\alpha})y(t)dt, \quad -1 \leqslant x \leqslant 1. \tag{3}$$

Here, $\alpha \in [0,1)$ and λ is a given constant. For $b = 0$, the integral equation (3) is a Volterra integral equation of the first kind and for $b \neq 0$ it is an equation of the second kind. Note that the kernel in (3) is (weakly) singular if $\alpha \in (0,1)$. The present method from its simplicity may be used as library program in computing laboratories for the approximate solutions of first and second kind Volterra integral equations with singular (weakly) and non-singular kernels.

In Chapter IV, we obtain two quadrature formulas for the numerical evaluation of weighted Cauchy principal value integral

$$I(f;a) = \fint_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}(f(x)/(x-a))dx, \qquad (4)$$

where $\alpha, \beta > -1$ and $a \in (-1,1)$. The first quadrature formula uses "practical" ~~"principal"~~ abscissas $x_k = \cos(k\pi/n)$, $k = 0(1)n$; as its "nodes" and the second quadrature formula uses "practical" abscissas $x_k = \cos(k\pi/n)$, $k = 0(1)n$ and 'a' as its "nodes". We establish the convergence of these quadrature formulas for a suitable class of functions. It should be noted here that we can also obtain quadrature formulas using "classical" abscissas $x_k = \cos((2k+1)\pi/(2n+2))$, $k = 0(1)n$, in place of "practical" abscissas as "nodes" in similar manner and these quadrature formulas will also be convergent for the same class of functions. Later on in Chapter V, we use the first quadrature formula, for obtaining the approximate solutions of the singular integral equations:

$$a\varphi(x) + (b/\pi) \fint_{-1}^{1} (\varphi(t)/(t-x))dt + \int_{-1}^{1} k(x,t)\varphi(t)dt$$

$$= g(x), \quad -1 < x < 1, \qquad (5)$$

where a and b are real constants k(x,t) and g(x) are Hölder-continuous functions. Equation (5) accepts a solution of the form:

$$\varphi(t) = w(t)f(t) \tag{6}$$

where

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}; \quad \alpha, \beta > -1.$$

After obtaining the approximate values of f on "practical" and "classical" abscissas we obtain the approximate values of the first derivative of f at the "practical" and "classical" abscissas (except at ± 1) by using second quadrature formula.

The thesis consists of five chapters and a brief description of the contents of each chapter follows:

Chapter I. Noble (1969) has described a method for the numerical solution of second kind Volterra integral equation (1) treating the nonlinear kernel as a single function and replacing the integral in each interval by repeated Simpson's and $\frac{3}{8}$-rules. We modify this method by replacing the unknown solution in each interval by repeated quadratic and cubic interpolating polynomials. The modified method is fourth order convergent with precision three (i.e., the method is exact if the unknown solution is a polynomial of degree less or equal to three) and has improved accuracy. We also describe a fourth order self-starting method based on the above concept; this method has precision one. Numerical examples are considered to demonstrate computationally the fourth order convergence of the modified method and the improved accuracy of the modified method over the method described in Noble (1969). This chapter is going to appear in BIT.

Chapter II. We describe second, third and fourth order self-starting explicit methods for the approximate solutions of (1) by treating unknown solution y(x) separately. These methods are exact if y(x) is a constant. We also describe a fourth order explicit method based on the above concept. This method requires starting values and is exact if y(x) is a polynomial of degree less or equal to two. Numerical examples are considered to show that the accuracies of the methods are improved by treating unknown solution y(x) separately. The self-starting fourth order method of this chapter is going to appear in BIT.

Chapter III. We describe a method for the approximate solutions of Volterra integral equation (3). The method consists in approximating y(x) by a truncated Chebyshev series. The approximate values of the Chebyshev-Fourier coefficients are found by solving the linear system obtained by discretization of the integral equation (3) at the "classical" abscissas. If the value of the solution is desired at the particular point in [-1,1], the finite Chebyshev series, with coefficients as determined above, can be summed up by the using Clenshaw's algorithm (1955). We also represent the known function $K(x,t)$, for fixed x, in terms of truncated Chebyshev series. Here we also assume that $y(x)$ and $K(.,t)$ are continuous and of bounded variation on $[-1,1]$. Numerical examples of Volterra integral equations of the first and second kinds are given to illustrate the method; these examples show that the present method gives quite satisfactory approximations for the Chebyshev-Fourier coefficients of the unknown solution $y(x)$. This chapter is published in Journal of Mathematical and Physical Sciences, 12 (1978), 473-481.

Chapter IV. We describe two quadrature formulas for the numerical evaluation of Cauchy principal value integral (4). First quadrature formula is obtained by replacing f by a polynomial interpolating f on "practical" abscissas $x_k = \cos(k\pi/n)$, $k = 0(1)n$. The second quadrature formula is obtained by subtracting out the singularity and uses "practical" abscissas and pole as its "nodes". We establish the convergence of the first quadrature formula for the class of functions which are Hölder-continuous on $[-1,1]$. We also show that the second quadrature formula is convergent if f satisfies the following conditions:

(i) $f \in H_\mu$, for $0 < \mu \leqslant 1$,

(ii) $f'(x)$ exists and continuous in a small neighbourhood of a,

(iii) $f''(a)$ exists.

The first quadrature formula of this chapter is published in Computing 23 (1979), 67-72.

Chapter V. In this chapter we describe numerical methods for the approximate solutions of the singular integral equation (5), which accepts a solution of the form

$$\phi(t) = w(t)f(t)$$

where

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}$$

and the constants α and β are given by:

$$\alpha = (2\pi i)^{-1} \log((a-ib)/(a+ib))+N, \quad \alpha > -1$$

$$\beta = -(2\pi i)^{-1} \log((a+ib)/(a+ib))+M, \quad \beta > -1$$

with N and M integers such that the index

$$\nu = -(\alpha+\beta) = -(N+M)$$

is restricted to the values -1, 0 and 1.

Approximate methods described by Krenk (1975,1978) for the numerical solution of (5) need zeros of the appropriate Jacobi polynomials; this requires sufficient amount of time and is also tedious.

Our aim in this chapter is to describe methods which provide approximate values of f on $[-1,1]$, including points ± 1 and also the above tedious job is removed. The above things may be accomplished by replacing f by a polynomial interpolating f on "practical" or "classical" abscissas. The approximate values of the unknown solution $f(x)$ at the "practical" and "classical" abscissas are obtained by solving the linear system obtained by discretization of the integral equation (5) at the zeros of the Chebyshev polynomials of the second kind with the extra condition (if it is there).

After obtaining the approximate values of f on "practical" and "classical" abscissas we obtain the approximate values of f' on "practical" and "classical" abscissas (except at ± 1) by using the second quadrature formula of Chapter IV.

Note: After submission of the synopsis, we have generalised the concepts of Chapters II and III. In Chapter II, we modify the modified increment methods of Garey (1975) for nonlinear second kind Volterra integral equations (1) by treating the unknown solution separately and have shown that the Garey's methods are particular cases of the present methods. Our modification improves the accuracy of the Garey's methods and also preserves the strong convergence and stability properties. In Chapter III, we describe a recurrence relation for the evaluation of integrals

$$I_{k,p}(x,\alpha) = \int_0^x \frac{T_k^*(t)}{(x^p-t^p)^\alpha}\,dt, \quad 0 \leqslant x \leqslant 1,$$

where p is a positive integer $\geqslant 1$, $0 < \alpha < 1$ and $T_k^*(t) = T_k(2t-1)$, $0 \leqslant t \leqslant 1$; ($T_k(t) = \cos(k \arccos t)$ be the Chebyshev polynomials of the first kind of degree k defined over $[-1, 1]$). The recurrence relation is useful in obtaining numerical methods for singular Volterra integral equations

$$by(x) = f(x) + \int_0^x (K(x, t, y(t))/(x^p - t^p)^\alpha)\,dt, \quad 0 \le x \le T,$$

using Chebyshev polynomials. The recurrence relation may also be used for $p = 1$ and $\alpha = 0$.

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