

**PARAMETER-UNIFORM NUMERICAL METHODS FOR  
SOME SINGULARLY PERTURBED  
CONVECTION-DIFFUSION PROBLEMS**

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# PARAMETER-UNIFORM NUMERICAL METHODS FOR SOME SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS

by

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Department of Mathematics

Submitted

*in fulfillment of the requirements of the degree of Doctor of Philosophy*

*to the*



Indian Institute of Technology Delhi  
October 2017

*To*  
*My Family*

# Certificate

This is to certify that the thesis entitled **Parameter-Uniform Numerical Methods for Some Singularly Perturbed Convection-Diffusion Problems** submitted by **Ms. Varsha Srivastava** to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by her under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi  
October 2017

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*"Miles to go before I sleep"*

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# Abstract

The solution of singular perturbation convection-diffusion problems contain layers and the width of the layers varies with respect to the perturbation parameter, due to this these problems can not be solved using classical methods on uniform meshes and so in this thesis we are interested in parameter-uniform numerical methods that work for all values of the perturbation parameter for some singularly perturbed convection-diffusion problems.

An almost second order uniformly convergent alternate overlapping Schwarz method is developed for solving singularly perturbed problems of types: linear convection-diffusion problems, a system of  $M(\geq 2)$  weakly coupled linear convection-diffusion problems and semilinear convection-diffusion problems.

A high order numerical method for a system of two weakly coupled singularly perturbed convection-diffusion problems multiplied with the distinct perturbation parameters is designed and the method yields almost second order parameter-uniform convergence with respect to the distinct perturbation parameters. A priori bounds on the solution, its derivatives and its decomposition into regular and singular components are also derived.

An alternate overlapping Schwarz method is constructed for time-dependent singularly perturbed problems of types: linear convection-diffusion problems, a weakly coupled systems of  $M(\geq 2)$  convection-diffusion problems and semilinear convection-diffusion problems using the backward Euler method in time direction and HODIE scheme in space direction. The numerical approximations obtained from these method are first order in time and almost second order in space.

Finally, two uniformly convergent numerical methods are designed for two weakly coupled systems of time-dependent singularly perturbed linear convection-diffusion problems. The designed Method-I is parameter-robust of first order in time and parameter-robust of almost first order in space; while the designed Method-II is parameter-robust of first order in time and parameter-robust of almost second order in space. We also analyze the behavior and a priori bounds of the solution, its derivatives and construct a suitable decomposition into the regular and singular components.

Numerical experiments are conducted on some standard examples for all the developed numerical schemes to validate the theoretical results.

## सार

एकवचन गड़बड़ी संवहन-प्रसार समस्याओं के समाधान में परतें शामिल हैं और परतों की चौड़ाई गड़बड़ी पैरामीटर के संबंध में भिन्न होती है, कारण इन समस्याओं को एक समान मेष पर शास्त्रीय तरीकों का उपयोग करके हल नहीं किया जा सकता

और इसलिए इस थीसिस में हम पैरामीटर-समान संख्यात्मक तरीकों में रुचि रखते हैं कुछ विचित्र रूप से घबराहट के लिए परेशान पैरामीटर के सभी मूल्यों के लिए काम करते हैं।

संवहन-प्रसार समस्याओं

लगभग दूसरे ऑर्डर समान रूप से अभिसरण वैकल्पिक ओवरलैपिंग श्वार्ज़

विधि प्रकार की समस्याओं का एकमुश्त परेशान करने के लिए विकसित की गई है: रैखिक

संवहन-प्रसार समस्याओं, एम ( $\geq 2$ ) की एक प्रणाली कमजोर युग्मित रैखिक संवहन

समस्याओं और अर्धसैनिक संवहन-प्रसार समस्याओं

कमजोर रूप से युग्मित दो प्रणालियों की एक प्रणाली के लिए उच्च क्रम संख्यात्मक पद्धति का विलक्षण रूप से घबराहट

संवहन-प्रसार समस्याओं को अलग-थलगता पैरामीटर के साथ गुणा किया जाता है

डिज़ाइन किया गया है और विधि लगभग दूसरे क्रम पैरामीटर-वर्दी का उत्पादन करती है।

विशिष्ट गड़बड़ी पैरामीटर के संबंध में कनवर्जेन्स एक प्राथमिकता सीमा

समाधान पर, इसके डेरिवेटिव और नियमित और एकवचन घटकों में इसकी अपघटन

भी व्युत्पन्न हैं।

एक वैकल्पिक ओवरलैपिंग श्वार्ज पद्धति का निर्माण समय-समय पर निर्भरता के लिए किया जाता है

प्रकार की परेशान समस्याओं: रेखिक संवहन-प्रसार समस्याओं, एक कमजोर रूप से युग्मित सिस्टम ऑफ एम ( $\geq 2$ ) संवहन-प्रसार समस्याओं और अर्धसैनिक संवहन कण समय की दिशा और HODIE में पिछड़े यूजर विधि का उपयोग करते हुए समस्याएं अंतरिक्ष की दिशा में योजना इन से प्राप्त संख्यात्मक अनुमान

विधि समय में पहली ऑर्डर होती है और अंतरिक्ष में लगभग दूसरी ऑर्डर होती है।

अंत में, दो समान रूप से संक्रमित संख्यात्मक तरीकों को कमजोर रूप से दो के लिए डिज़ाइन किया गया है।

युग्मित समय-परित्यक्त प्रणालियों की एकमात्र उलझन में रेखीय संवहन-प्रसार समस्याएँ। डिज़ाइन किए गए विधि-। समय-समय पर प्रथम आदेश के पैरामीटर-मजबूत है पैरामीटर-अंतरिक्ष में लगभग पहले आदेश के मजबूत; जबकि डिज़ाइन विधि ॥ है समय-समय पर प्रथम क्रम के पैरामीटर-मजबूत और लगभग दूसरे क्रम के मजबूत-पैरामीटर अंतरिक्ष में। हम समाधान के व्यवहार और पूर्व की सीमा का भी विश्लेषण करते हैं, इसके डेरिवेटिव और नियमित और एकवचन में उपयुक्त अपघटन का निर्माण अवयव।

सभी के लिए कुछ मानक उदाहरणों पर संख्यात्मक प्रयोग किया जाता है

सैद्धांतिक परिणामों को मान्य करने के लिए संख्यात्मक योजनाएं विकसित की हैं।

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# List of Symbols

$\varepsilon$	singular perturbation parameter
$h$	step size
$k$	iteration number
$N, \Delta t$	mesh discretization parameter
$\Omega$	space variable domain
$\partial\Omega$	boundary of $\Omega$
$\sigma$	mesh transition parameter
$\Omega_p, p = l, r$	subdomains of $\Omega$
$\bar{\Omega}^N$	$\bar{\Omega}_r^N \cup (\bar{\Omega}_l^N \setminus \bar{\Omega}_r^N)$
$X = \Omega \times \omega$	domain for non-stationary problems
$X_p, p = l, r$	subdomains of $X$
$\bar{X}^{N, N_t}$	$\bar{X}_r^{N, N_t} \cup (\bar{X}_l^{N, N_t} \setminus \bar{X}_r^{N, N_t})$
$C^s(\Omega), C^{s,n}(X)$	function spaces
$\mathbf{z}$	$(z_1, \dots, z_M)^T$
$z_j$	$z(x_j)$
$z_{p,j}$	$z_p(x_j)$
$\mathbf{z}_j$	$\mathbf{z}(x_j) = (z_{1,j}, \dots, z_{M,j})^T$

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$\mathbf{z}_{p,j}$	$\mathbf{z}_p(x_j) = (z_{p,1,j}, \dots, z_{p,M,j})^T$
$\mathbf{z} \leq \mathbf{w}$	if $z_i \leq w_i, 1 \leq i \leq M$
$ \mathbf{z} $	$( z_1 , \dots,  z_M )^T$
$\ z\ _D$	$\max_{x \in D}  z(x) , D \subset \Omega$
$\ \mathbf{z}\ _D$	$\max\{\ z_1\ _D, \dots, \ z_M\ _D\}$
$\ \cdot\ $	when domain is obvious, or of no particular significance
$\ \mathbf{z}\ _\star$	$\max_{1 \leq i \leq M}  z_i $
$\ z\ _{\bar{\Omega}^N}$	$\max_{x_j \in \bar{\Omega}^N}  z(x_j) $
$\ \mathbf{z}\ _{\bar{\Omega}^N}$	$\max\{\ z_1\ _{\bar{\Omega}^N}, \dots, \ z_M\ _{\bar{\Omega}^N}\}$
$\ z\ _{\bar{X}^{N,N_t}}$	$\max_{(x_j, t_n) \in \bar{X}^{N,N_t}}  z(x_j, t_n) $
$\ \mathbf{z}\ _{\bar{X}^{N,N_t}}$	$\max\{\ z_1\ _{\bar{X}^N}, \dots, \ z_M\ _{\bar{X}^N}\}$
$C \ C_n \ n = 0, 1, \dots$	generic positive constants, independent of $\varepsilon, N, k,$ and $\Delta t$
$\mathbf{C} \ \mathbf{C}_n \ n = 0, 1, \dots$	generic positive constant vector
$\rho_\varepsilon^N$	classical rate of convergence for fixed $\varepsilon$
$\rho^N$	parameter-robust rate of convergence for all $\varepsilon$
$E_\varepsilon^N$	maximum error at nodal points for fixed $\varepsilon$
$E^N$	$\max_\varepsilon E_\varepsilon^N$

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