GROUP THEORETIC TECHNIQUES FOR THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

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DEDICATED TO

MY FATHER, MOTHER AND BROTHERS

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CERTIFICATE

This is to certify that the thesis entitled:

'GROUP THEORETIC TECHNIQUES FOR THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS',

which is being submitted by Mr. K. Vijayakumar, Research scholar, Mathematics Department to the Indian Institute of Technology, Delhi, for the award of the DEGREE OF DOCTOR OF PHILOSOPHY in MATHEMATICS, is a record of bonafide research work carried out by him under my guidance and supervision and has fulfilled all the requirements for the submission of this thesis.

The results contained in this thesis have not been submitted in part or full, to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

The thesis entitled "Group theoretic techniques for the solutions of nonlinear differential equations of physical systems" is comprised of eight chapters. Chapter I is devoted to the study of the various group theoretic techniques for solving nonlinear differential equations of physical systems and the survey of the works related to the work put up in Chapters II-VIII. Chapters II-IV deal with the applications of isovector approach to study, respectively, the solutions of a class of 'Generalised nonlinear diffusion equations', 'Semilinear hyperbolic equations' and the equations of 'Dynamic meteorology'. In Chapters V-VII, we have carried over the so called 'Symmetry method' to determine the solutions of nonlinear differential equations corresponding to the 'Generalised Korteweg de Vries (K-dV) in a variable medium', '(p+1)th order Boltzmann equation and the generalised Mckean model of Boltzmann equation' and 'coupled reaction-diffusion equations'. Finally, in Chapter VIII, we have investigated the invariance and conservation laws for certain wave type equations. In the following sections, we briefly outline the details of each of the chapters.

CHAPTER-I: INTRODUCTION

As mentioned above, this chapter is devoted to the study of the various group theoretic techniques and a brief survey and developments of the works related to it. More specifically, we have, herein, given a brief resume of the isovector approach and symmetry method for determining the solutions of a single or a system of nonlinear partial differential equations.

Also included in this chapter are the necessary mathematical tools for establishing the existence of a functional for a single nonlinear differential equation, the method for writing down the functional when it exists and the procedure for the determination of one-parameter infinitesimal transformation via classical Noether's theorem with special reference to Rund's invariance identities.

CHAPTER-II: ON THE ISOGROUPS OF THE GENERALISED DIFFUSION EQUATION

Herein, the generalised diffusion equation which encompasses the nonlinear diffusion equation with a source term and the Boussinesq equation in hydrology as its particular form and appears in a wide variety of physical and engineering applications has been analysed via isovector approach. More specifically, we have first constructed isovector fields by translating the given diffusion equation in the language of exterior differential forms over a five-dimensional space. Further, using the transport property of the forms under Lie's derivative the components of the associated isovector field are constructed. This results in the orbital equations corresponding to the components of the isovector, which are solved to obtain invariant groups of transformations that eventually reduce the given nonlinear partial differential equation to nonlinear ordinary differential equation (NLODE). On solving the NLODE for physically realizable forms of the parameters involved, all the available results for particular forms of the diffusion equation are recovered and quite a number of new exact solutions are reported.

CHAPTER-III: ON CERTAIN NEW AND EXACT SOLUTIONS OF A CLASS OF SEMILINEAR HYPERBOLIC EQUATIONS VIA ISOVECTOR APPROACH AND NOETHER'S THEOREM

In this chapter, we have utilized isovector approach to obtain exact solutions of another important class of equations-semilinear hyperbolic equation $u_{xt} = f(u)$, where $f(u)$ is an arbitrary function. Following the same procedure as outlined in Chapter I, we have tabulated the components of the isovector for eight different forms of $f(u)$ and solved the corresponding orbital equations only for four physically interesting and different choices of $f(u)$. The solutions of the orbital equations lead us to the invariant group of transformations, which reduce the given nonlinear partial differential equations to nonlinear ordinary differential equations. Except for the choice $f(u) = e^{nu}$, $n \neq 0$, all the ordinary differential equations have either been solved through the usual available technique or reduced to standard form/Painleve form. When $f(u)$ is of the form e^{nu}, $n \neq 0$, the resulting nonlinear ordinary differential equation which is Liouville equation has been solved, via Noether's theorem. In particular, after establishing the existence of the functional and hence formulating it through the mathematical tools described in Chapter I, we have utilized the invariance identities of Rund involving the Lagrangian and the generators of the infinitesimal Lie group and Noether's theorems to obtain first integral of the Liouville equation. The repeated applications of the invariant transformation results in the exact solution of the first order equation which does not seem to have been reported earlier via any technique, whatsoever.

CHAPTER-IV: ON THE ISOGROUPS OF THE SYSTEM OF EQUATIONS OF DYNAMIC METEOROLOGY

Unlike Chapters II and III, herein the isovector approach is utilized for obtaining invariant solutions of the system of nonlinear partial differential equations governing the steady two dimensional flow of an inviscid fluid with Coriolis term. More specifically, solutions for different forms of the Coriolis parameter have been obtained using the similarity transformations that are constructed from isovectors and compared with the available results obtained via other group theoretic techniques.

CHAPTER V: ON THE INVARIANT SOLUTIONS OF THE GENERALISED KORTEWEG-de-VRIES EQUATION IN A VARIABLE MEDIUM

In this chapter, the invariance under continuous group of transformation of the generalised Korteweg-de-Vries equation in a variable medium has been studied via a generalised procedure called symmetry method. To this effect possible invariant solutions of equations for physically realizable form of variable coefficient and the function of dependent variable are obtained. This new procedure beside yielding new invariants has provided soliton like solutions for variable coefficient K-dV, modified K-dV and K-dV with higher order nonlinearity and other new exact solutions which have hitherto been unexplored.

CHAPTER VI: ON THE INVARIANT SOLUTIONS OF THE DIFFERENT MODELS OF BOLTZMANN EQUATION

In this chapter, we have carried over symmetry method to obtain invariant transformations to reduce the partial differential equations corresponding to the following three different models of Boltzmann equation to nonlinear ordinary differential equations:

(i) (p+1)th order model

(ii) Generalised form of Mckean model

(iii) Krook-Wu model

An interesting outcome of the study is the reporting of a generalised symmetry operator in the form of a conjecture for case (i) and the new exact solution for the resulting ordinary differential equation for case (ii), which includes Carleman model as its particular case. For case (iii), the resulting NLODE has been solved via Noether's theorem which necessitates reducing it to first integral by using the invariance identities of Rund involving the Lagrangian and its generators of the infinitesimal Lie group. Through the repeated application of the invariant transformation we have arrived at a new and exact solution of Krook-Wu equations.

CHAPTER-VII: ON INVARIANT SOLUTIONS OF SYSTEM OF COUPLED REACTION-DIFFUSION EQUATIONS

Unlike Chapters V-VI, where in we have taken up the application of symmetry method to investigate similarity solutions of single partial differential equations, herein we have carried over its application to a system of coupled nonlinear reaction-diffusion equations for the constant diffusion coefficients. Using this generalised procedure, a complete set of symmetries has been obtained for various choices of functions of the dependent variables occuring in the equations. Further, by using the symmetries, invariant transformations are constructed which leaves the system of equations invariant. Some new classes of similarity solutions are obtained which for a particular case yield the solution in terms of Kummer functions.

CHAPTER-VIII: INVARIANCE AND CONSERVATION LAWS FOR CERTAIN CLASSES OF WAVE TYPE EQUATIONS

Using Nœther's theorem, one-to-one correspondence is established between the conservation laws and infinitesimal transformations for the following classes of wave type equations:

(i) $u_{xt} = f(u)$, where $f(u)$ is an arbitrary function.

(ii) $u_{tt} + 2\beta u_{xt} - (1-\beta^2)u_{xx} - k\, u_x^2\, u_{xx} = 0$, where $k = \frac{3}{2\alpha^2}$,

α is the initial tension and β is the initial velocity of vibration of a moving string.

(iii) $u_{tt} = (a+bu_x^\alpha)u_{xx}$, where α is an integer ≥ 1.

As the availability of a Lagrangian for these equations is necessary for the applicability of Noether's theorem, we have thus, first proved the existence and hence formulated the functional whose stationarity yields the above partial differential equations.

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